SINGULARITIES OF EQUIDISTANTS AND GLOBAL CENTRE SYMMETRY SETS OF LAGRANGIAN SUBMANIFOLDS

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Abstract. We define the Global Centre Symmetry set (GCS) of a smooth closed m-dimensional submanifold $M \subset \mathbb{R}^n$, $n \leq 2m$, which is an affinely invariant generalization of the centre of a k-sphere in $\mathbb{R}^{k+1}$. The GCS includes both the centre symmetry set defined by Janeczko [15] and the Wigner caustic defined by Berry [3]. We develop a new method for studying generic singularities of the GCS which is suited to the case when M is Lagrangian in $\mathbb{R}^{2m}$ with canonical symplectic form. The definition of the GCS, which slightly generalizes one by Giblin and Zakalyukin [9]-[11], is based on the notion of affine equidistants, so, we first study singularities of affine equidistants of Lagrangian submanifolds, classifying all the stable ones. Then, we classify the affine-Lagrangian stable singularities of the GCS of Lagrangian submanifolds and show that, already for smooth closed convex curves in $\mathbb{R}^2$, many singularities of the GCS which are affine stable are not affine-Lagrangian stable.

1. Introduction

A circle is usually defined as the set of all points on a plane which are equidistant to a fixed point. Naturally, this point is called the centre of the circle or, equivalently, the centre of symmetry of the circle. And similarly for an n-sphere in $\mathbb{R}^{n+1}$. However, the above definition depends on a choice of metric in $\mathbb{R}^{n+1}$.

When trying to generalize the notion of the centre of symmetry of an n-sphere in $\mathbb{R}^{n+1}$, in an affine invariant way, one finds that there seems to be more than one way of doing it. Looking at a circle on the plane, or even an ellipse, its centre can be defined as the set (in this case consisting of a single element) of midpoints of straight lines connecting pairs of points on the curve with parallel tangent vectors.

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For a generic smooth convex closed curve, this set is not a single point, but forms a curve with an odd number of cusps, in the interior of the smooth original curve. This singular inner curve has been known as the *Wigner caustic* of a smooth closed curve since the work of Berry in the 70's, because of its prominent appearance in the semiclassical limit of the Wigner function of a pure quantum state whose classical limit corresponds to the given smooth curve in $\mathbb{R}^2$, with canonical symplectic structure (see [3] and [16] for more details).

Therefore, the Wigner caustic is a natural affine-invariant generalization of the centre of symmetry of a circle, or an ellipse, which extends to higher dimensional smooth closed submanifolds of $\mathbb{R}^n$.

On the other hand, the centre of a circle or an ellipse in $\mathbb{R}^2$ can also be described as the envelope of all straight lines connecting pairs of points on the curve with parallel tangent vectors.

For a generic smooth convex closed curve, this set is not a single point, but forms a curve with an odd number of cusps, in the interior of the smooth original curve. This singular inner curve has been known as the *centre symmetry set* (CSS) of a smooth closed curve since the work of Janeczko in the 90's and is a natural affine-invariant generalization of the centre of symmetry of a circle, or an ellipse, which extends to higher dimensional smooth closed submanifolds of $\mathbb{R}^n$ [15].

However, except for circles or ellipses, when both symmetry sets are the same point, the Wigner caustic and the centre symmetry set of a smooth convex closed curve are not the same singular curve. Instead, the Wigner caustic is interior to the centre symmetry set and the cusp points of the inner curve touches the outer one in its smooth part.

A new, more complicated curve, containing the Wigner caustic and the centre symmetry set, can be defined in a single way and this affine-invariant definition extends to an arbitrary smooth closed m-dimensional submanifold M of $\mathbb{R}^n$, for $n \leq 2m$. We call this new set the *Global Centre Symmetry* set of M, denoted by $GCS(M)$.

In fact, our definition is only a very slight modification of a definition already introduced and used by Giblin and Zakalyukin [9]-[11] to study singularities of centre symmetry sets of hypersurfaces. A key notion in their definition is that of an affine λ-equidistant to the smooth submanifold, of which the Wigner caustic is the case $\lambda = 1/2$. The singularities of these λ-equidistants are then fundamental to characterize the Global Centre Symmetry set and its singularities.

In this paper, we present a new method for studying the singularities of affine λ-equidistants $E_\lambda(L)$, $\forall \lambda \in \mathbb{R}$, and the affine-invariant $GCS(L)$ of a smooth closed *Lagrangian* submanifold L of affine symplectic space $(\mathbb{R}^{2n}, \omega)$, where ω is the canonical symplectic form.

This paper is organized as follows. In section 2 we present the definition of the Global Centre Symmetry set. This section also contains the basic definitions of degree of parallelism, affine equidistant, Wigner caustic, centre symmetry caustic and criminant. In section 3 we define λ-chord transformations which are used to define a general characterization and classification for affine equidistants.

In section 4 we define the generating families for these affine equidistants and relate their general classification to the well known classification by Lagrangian equivalence [2]. This is used in section 5 to obtain the classification of stable singularities of affine equidistants of Lagrangian submanifolds. Theorem 5.1 states that any caustic of stable Lagrangian singularity is realizable as $E_\lambda(L)$, for some Lagrangian $L \subset (\mathbb{R}^{2m}, \omega)$, and Corollaries 5.2 and 5.3 specialize this theorem to the cases when L is a curve or a surface. In the first case, generic singularities are cusps, while, in the second case, they can be cusps, swallowtails, butterflies, or hyperbolic, elliptic and parabolic umbilics.

The following three sections are devoted to the singularities of the Global Centre Symmetry set. In section 6 we give a geometric characterization for the criminant of $GCS(L)$ similar to results in [9]-[11] for hypersurfaces. In section 7 we introduce the equivalence relation (also as an equivalence of generating families) that allows for a complete affine-symplectic-invariant classification of the stable singularities of $GCS(L)$. We show that only singularities of the criminant, the smooth part of the Wigner caustic, or tangent union of both, are stable.

Finally, section 8 is devoted to the study of the GCS of Lagrangian curves. First, we state two theorems for the GCS of convex curves in $\mathbb{R}^2$ when no symplectic structure is considered. The results presented in Theorem 8.1 are not new ([3], [15], [8]-[12]), but, in Theorem 8.2 the inequality on the number of cusps of the CSS and the Wigner caustic, although straightforward from the results in [8], had not been mentioned before. Pictures illustrate these theorems. Then, we specialize the results of section 7 to the case of Lagrangian curves, showing that most of the singularities which were affine-stable when no symplectic structure was considered are not affine-Lagrangian stable.

In other words, although any smooth curve on $\mathbb{R}^2$ is Lagrangian, the singularities of their GCS are sensitive to the presence of a symplectic form to be accounted for, that is, there is a breakdown of the stability of many of these singularities. This is similar to some results in [4]-[7].

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2. Definition of the Global Centre Symmetry set.

Let M be a smooth closed m-dimensional submanifold of the affine space $\mathbb{R}^n$, with $n \leq 2m$. Let a, b be points of M. Let τ_{a-b} be the translation by the vector $(a-b)$

$$\tau_{a-b} : \mathbb{R}^n \ni x \mapsto x + (a-b) \in \mathbb{R}^n.$$

Definition 2.1. A pair of points $a, b \in M$ $(a \neq b)$ is called a **weakly parallel** pair if

$$T_aM + \tau_{a-b}(T_bM) \neq T_a\mathbb{R}^n.$$

$\operatorname{codim}(T_aM + \tau_{a-b}(T_bM))$ in $T_a\mathbb{R}^n$ is called a **codimension of a weakly parallel pair** a, b. We denote it by $\operatorname{codim}(a, b)$.

A weakly parallel pair $a, b \in M$ is called k**-parallel** if

$$\dim(T_aM \cap \tau_{b-a}(T_bM)) = k.$$

If $k = m$ the pair $a, b \in M$ is called **strongly parallel**, or just **parallel**. We also refer to k as the **degree of parallelism** of the pair (a, b) and denote it by $\deg(a, b)$. The degree of parallelism and the codimension of parallelism are related in the following way:

$$2m - \deg(a, b) = n - \operatorname{codim}(a, b). \tag{2.1}$$

Thus, *for a Lagrangian submanifold, the degree of parallelism and the codimension of a weakly parallel pair coincide.*

Definition 2.2. A **chord** passing through a pair a, b, is the line

$$l(a, b) = \{x \in \mathbb{R}^n | x = \lambda a + (1-\lambda)b, \lambda \in \mathbb{R}\},$$

but we sometimes also refer to $l(a, b)$ as a chord *joining* a and b.

Definition 2.3. For a given λ, an **affine** λ**-equidistant** of M, $E_\lambda(M)$, is the set of all $x \in \mathbb{R}^n$ such that $x = \lambda a + (1-\lambda)b$, for all weakly parallel pairs $a, b \in M$. $E_\lambda(M)$ is also called a (affine) **momentary equidistant** of M. Whenever M is understood, we write E_λ for $E_\lambda(M)$.

Note that, for any λ, $E_\lambda(M) = E_{1-\lambda}(M)$ and in particular $E_0(M) = E_1(M) = M$. Thus, the case $\lambda = 1/2$ is special:

Definition 2.4. $E_{\frac{1}{2}}(M)$ is called the **Wigner caustic** of M.

Remark 2.5. This name is given for historical reasons [3], [16].

The *extended affine space* is the space $\mathbb{R}^{n+1}_e = \mathbb{R}\times\mathbb{R}^n$ with coordinate $\lambda \in \mathbb{R}$ (called *affine time*) on the first factor and projection on the second factor denoted by $\pi : \mathbb{R}^{n+1}_e \ni (\lambda, x) \mapsto x \in \mathbb{R}^n$.

Definition 2.6. The **affine extended wave front** of M, $\mathbb{E}(M)$, is the union of all affine equidistants each embedded into its own slice of the extended affine space: $\mathbb{E}(M) = \bigcup_{\lambda\in\mathbb{R}} \{\lambda\} \times E_\lambda(M) \ \subset \mathbb{R}^{n+1}_e$.

Note that, when M is a circle on the plane, $\mathbb{E}(M)$ is the (double) cone, which is a smooth manifold with nonsingular projection π everywhere, but at its singular point, which projects to the centre of the circle. From this, we generalize the notion of centre of symmetry.

Thus, let π_r be the restriction of π to the affine extended wave front of M: $\pi_r = \pi|_{\mathbb{E}(M)}$. A point $x \in \mathbb{E}(M)$ is a **critical** point of π_r if the germ of π_r at x fails to be the germ of a regular projection of a smooth submanifold. We now introduce the main definition of this paper:

Definition 2.7. The **Global Centre Symmetry** set of M, $GCS(M)$, is the image under π of the locus of critical points of π_r.

Remark 2.8. The set $GCS(M)$ is the bifurcation set of a family of affine equidistants (family of chords of weakly parallel pairs) of M.

Remark 2.9. In general, $GCS(M)$ consists of two components: the **caustic** $\Sigma(M)$ being the projection of the singular locus of $\mathbb{E}(M)$ and the **criminant** $\Delta(M)$ being the (closure of) the image under π_r of the set of regular points of $\mathbb{E}(M)$ which are critical points of the projection π restricted to the regular part of $\mathbb{E}(M)$. $\Delta(M)$ is the envelope of the family of regular parts of momentary equidistants, while $\Sigma(M)$ contains all the singular points of momentary equidistants.

The above definition (with its following remarks) is only a very slight modification of the definition that has already been introduced and used by Giblin and Zakalyukin [9] to study centre symmetry sets of hypersurfaces. However, in our present definition the whole manifold M is considered, as opposed to pairs of germs, as in [9], and weak parallelism is also taken into account. Considering the whole manifold in the definition leads to the following simple but important result:

Theorem 2.10. *The Global Centre Symmetry set of M contains the Wigner caustic of M.*

Proof. Let x be a regular point of $E_{\frac{1}{2}}(M)$. Then $x = \frac{1}{2}(a+b)$ for a weakly parallel pair $a, b \in M$. It means that x is a intersection point of the chords $l(a,b)$ and $l(b,a)$. Then $\mathbb{E}(M)$ contains the sets

$$\{(\lambda, \lambda a + (1-\lambda)b) | \lambda \in \mathbb{R}\},\ \{(\lambda, (1-\lambda)a + \lambda b) | \lambda \in \mathbb{R}\}.$$

If $(\frac{1}{2}, x)$ is a regular point of $\mathbb{E}(M)$ then the above sets are included in the tangent space to $\mathbb{E}(M)$ at $(\frac{1}{2}, x)$. It implies that a fiber $\{(\lambda, x)|\lambda \in \mathbb{R}\}$ is included in the tangent space of $\mathbb{E}(M)$. Thus if $(\frac{1}{2}, x)$ is a regular point of $\mathbb{E}(M)$ then x is in the criminant $\Delta(M)$. If $(\frac{1}{2}, x)$ is not a regular point of $\mathbb{E}(M)$ then x is in the caustic $\Sigma(M)$. □

Remark 2.11. From the proof of the previous theorem, it is clear that the Wigner caustic belongs to the self-intersection set of $\mathbb{E}(M)$, so it is a part of the caustic $\Sigma(M)$. In view of this fact, we divide $\Sigma(M)$ into two parts: The Wigner caustic $E_{1/2}(M)$ and the **centre symmetry caustic** $\Sigma'(M) = \Sigma(M) \setminus E_{1/2}(M)$.

Remark 2.12. In the literature, if $M \subset \mathbb{R}^2$ is a smooth curve, $E_{1/2}(M)$ has been described in various ways. First, $E_{1/2}(M) \ni x$ is the bifurcation set for the number of chords connecting two points in M given a chord midpoint $x \in \mathbb{R}^2$ [3]. Similarly, if $\mathcal{R}_x : \mathbb{R}^2 \to \mathbb{R}^2$ denotes reflection through $x \in \mathbb{R}^2$, then $x \in E_{1/2}(M)$ when M and $\mathcal{R}_x(M)$ are not transversal [16]. Finally, considering the area of the region enclosed by M and a chord as a function A of a point x on the chord and a variable κ locating one of the endpoints of the chord on the curve, then, regarding x as parameter, $A(x, \kappa)$ is a generating family for which $E_{1/2}(M)$ is its bifurcation set [3, 12]. This third description is generalized below in section 4 to every λ-equidistant of any Lagrangian submanifold.

3. λ-CHORD TRANSFORMATIONS

For $\lambda = 1/2$, there is a well known procedure, sometimes known as the centre-chord change of coordinates, sometimes as the midpoint transformation, hereby also called the "$\frac{1}{2}$-chord transformation", which encodes the midpoint reflections referred to in Remark 2.12 above, in such a way as to facilitate the description of the Wigner caustic [18].

Consider the product affine space: $\mathbb{R}^n \times \mathbb{R}^n$ with coordinates (x_+, x_-) and the tangent bundle to $\mathbb{R}^n$: $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ with coordinate system $(x, \dot{x})$ and standard projection $pr : T\mathbb{R}^n \ni (x, \dot{x}) \to x \in \mathbb{R}^n$. Then, there exists a global linear diffeomorphism

$$\Phi_{1/2} : \mathbb{R}^n \times \mathbb{R}^n \ni (x^+, x^-) \mapsto \left(\frac{x^+ + x^-}{2}, \frac{x^+ - x^-}{2}\right) = (x, \dot{x}) \in T\mathbb{R}^n,$$

$$\Phi_{1/2}^{-1} : T\mathbb{R}^n \ni (x, \dot{x}) \mapsto (x + \dot{x}, x - \dot{x}) = (x^+, x^-) \in \mathbb{R}^n \times \mathbb{R}^n.$$

This map $\Phi_{1/2}$ is the $\frac{1}{2}$-chord transformation, which we now generalize.

Definition 3.1. $\forall \lambda \in \mathbb{R} \setminus \{0, 1\}$, a **$\lambda$-chord transformation**

$$\Phi_\lambda : \mathbb{R}^n \times \mathbb{R}^n \to T\mathbb{R}^n \ , \ (x^+, x^-) \mapsto (x, \dot{x})$$

is a *linear* diffeomorphism generalizing the half-chord transformation, which is defined by the λ-*point equation*:

$$x = \lambda x^{+} + (1-\lambda)x^{-} \ , \tag{3.1}$$

for the λ-point x, and the *chord equation*:

$$\dot{x} = \lambda x^{+} - (1-\lambda)x^{-}, \tag{3.2}$$

with the inverse map Φ_λ^{-1} given by:

$$x^{+} = \frac{x+\dot{x}}{2\lambda} \quad , \quad x^{-} = \frac{x-\dot{x}}{2(1-\lambda)} \ . \tag{3.3}$$

Now, let M be a smooth closed m-dimensional submanifold of the affine space $\mathbb{R}^n$ $(2m \geq n)$ and consider the product $M \times M \subset \mathbb{R}^n \times \mathbb{R}^n$. Let $\mathcal{M}_\lambda$ denote the image of $M \times M$ by a λ-chord transformation,

$$\mathcal{M}_\lambda = \Phi_\lambda(M \times M) \ ,$$

which is a $2m$-dimensional smooth submanifold of $T\mathbb{R}^n$.

Then we have the following general characterization:

Theorem 3.2. *The set of critical values of the standard projection* $pr : T\mathbb{R}^n \to \mathbb{R}^n$ *restricted to* $\mathcal{M}_\lambda$ *is* $E_\lambda(M)$.

Proof. Let a belong to the set of critical values of $pr|_{\mathcal{M}_\lambda}$. It means that $\dim T_{(a,\dot{a})}\mathcal{M}_\lambda \cap T_{(a,\dot{a})}pr^{-1}(a)$ is greater than $2m-n$. Let $v_1, \cdots, v_k$ for $k > 2m-n$ be a basis of $T_{(a,\dot{a})}\mathcal{M}_\lambda \cap T_{(a,\dot{a})}pr^{-1}(a)$. Then these basis has the following form $v_j = \sum_{i=1}^n \alpha_{ji}\frac{\partial}{\partial \dot{x}_i}|_{(a,\dot{a})}$ for $j = 1, \cdots, k$. By (3.3) we get that $(\Phi_\lambda^{-1})_*(v_j) = \frac{1}{2\lambda}v_j^{+} - \frac{1}{2(1-\lambda)}v_j^{-}$, where

$$v_j^{+} = \sum_{i=1}^n \alpha_{ji}\frac{\partial}{\partial x_i^{+}}|_{a^{+}} \in T_{a^{+}}M, \quad v_j^{-} = \sum_{i=1}^n \alpha_{ji}\frac{\partial}{\partial x_i^{-}}|_{a^{-}} \in T_{a^{-}}M.$$

It implies that $v_j^{+} \in T_{a^{+}}M \cap \tau_{(a^{+}-a^{-})}T_{a^{-}}M$ for $j = 1, \cdots, k$. Since $k > 2m-n$ then $T_{a^{+}}M + \tau_{(a^{+}-a^{-})}T_{a^{-}}M \neq T_{a^{+}}\mathbb{R}^n$ and consequently a^{+}, a^{-} is a weakly parallel (k-parallel) pair. Hence $a = \lambda a^{+} + (1-\lambda)a^{-}$ belongs to E_λ.

Now assume that a belongs to E_λ. Then $a = \lambda a^{+} + (1-\lambda)a^{-}$ for a weakly k-parallel pair a^{+}, a^{-} for $k > 2m-n$. Thus there exist linearly independent vectors $v_j^{+} = \sum_{i=1}^n \alpha_{ji}\frac{\partial}{\partial x_i^{+}}|_{a^{+}} \in T_{a^{+}}M \cap \tau_{(a^{+}-a^{-})}T_{a^{-}}M$ for $j = 1, \cdots, k$. Consider linearly independent vectors $v_j = (\Phi_\lambda)_*((1-\lambda)v_j^{+} - \lambda\tau_{(a^{-}-a^{+})}v_j^{+})$ for $j = 1, \cdots, k$. It is obvious that v_j belongs to $T_{(a,\dot{a})}\mathcal{M}_\lambda$ and $pr_*(v_j) = 0$ for $j = 1, \ldots, k$. Thus a is a critical value of $pr|_{\mathcal{M}_\lambda}$. □

4. Generating families

Let $(\mathbb{R}^{2m}, \omega)$ be the affine symplectic space with canonical Darboux coordinates p_i, q_i, so that $\omega = \sum_{i=1}^{m} dp_i \wedge dq_i$, and let L be a smooth closed Lagrangian submanifold of $(\mathbb{R}^{2m}, \omega)$.

The purpose of this work is to describe the singularities of $GCS(L)$. To do so, we generalize to any $\lambda \in \mathbb{R}\backslash\{0,1\}$ another construction that is well known for $\lambda = 1/2$ (for this case, see for instance [18]). This other generalization amounts to correctly weighting the symplectic form on each copy of $\mathbb{R}^{2m}$ to be consistent with λ-chord transformations.

Thus, for a fixed $\lambda \in \mathbb{R} \setminus \{0,1\}$ we consider the product affine space $\mathbb{R}^{2m} \times \mathbb{R}^{2m}$ with the λ-weighted symplectic form

$$\delta_\lambda \omega = 2\lambda^2 \pi_1^* \omega - 2(1-\lambda)^2 \pi_2^* \omega \ , \tag{4.1}$$

where π_i is the projection of $\mathbb{R}^{2m} \times \mathbb{R}^{2m}$ on i-th factor for $i = 1, 2$.

Now, let Φ_λ be the λ-chord transformation (3.1)(3.2). Then,

$$\left(\Phi_\lambda^{-1}\right)^* (\delta_\lambda \omega) \ = \ \dot{\omega} \ . \tag{4.2}$$

where $\dot{\omega}$ is the canonical symplectic form on the tangent bundle to $(\mathbb{R}^{2m}, \omega)$, which is defined by $\dot{\omega}(x, \dot{x}) = d\{\dot{x} \lrcorner \omega\}(x)$ or, in Darboux coordinates for ω, by

$$\dot{\omega} = \sum_{i=1}^{m} d\dot{p}_i \wedge dq_i + dp_i \wedge d\dot{q}_i \ . \tag{4.3}$$

The pair $(T\mathbb{R}^{2m}, \dot{\omega})$ is the *canonical symplectic tangent bundle* of $(\mathbb{R}^{2m}, \omega)$ and is, thus, a $4m$-dimensional symplectic space.

The fibers of $T\mathbb{R}^{2m}$ are Lagrangian for $\dot{\omega}$, which means that $pr : T\mathbb{R}^{2m} \to \mathbb{R}^{2m}$ defines a **Lagrangian fiber bundle** with respect to $\dot{\omega}$, that is, a fiber bundle whose total space is equipped with a symplectic structure and whose fibers are Lagrangian submanifolds [2]. We let

$$\mathcal{L}_\lambda = \Phi_\lambda(L \times L).$$

Remark 4.1. In order to understand the ideology of this present construction, let's first focus attention on the case $\lambda = 1/2$ and consider a Lagrangian submanifold $\Lambda \subset (\mathbb{R}^{2m} \times \mathbb{R}^{2m}, \delta_{1/2}\omega)$ that projects regularly onto the first factor of $\mathbb{R}^{2m} \times \mathbb{R}^{2m}$, so that Λ is the graph of a symplectomorphism, or a *canonical transformation* $\psi : (\mathbb{R}^{2m}, \omega) \to (\mathbb{R}^{2m}, \omega)$, $\psi^* \omega = \omega$. If $\mathcal{L}_{1/2} = \Phi_{1/2}(\Lambda)$ locally projects regularly to the zero section of $(T\mathbb{R}^{2m}, \dot{\omega})$, then this canonical transformation $\psi : x^- \mapsto x^+$, can locally be "described" by the midpoint $x = (x^+ + x^-)/2$, that is, can locally be described by a *generating function* of the midpoints. Such a midpoint description was first introduced by Poincaré [17].

This generalizes for when Λ does not projects regularly to the first factor of $(\mathbb{R}^{2m} \times \mathbb{R}^{2m}, \delta_{1/2}\omega)$, but is still Lagrangian. In this case, Λ defines a *canonical relation* on $(\mathbb{R}^{2m}, \omega)$ and, if $\mathcal{L}_{1/2}$ locally projects regularly to the zero section of $(T\mathbb{R}^{2m}, \dot{\omega})$, this canonical relation can also be locally described by a generating function of the midpoints [18].

Clearly, if L is Lagrangian in $(\mathbb{R}^{2m}, \omega)$, then $L \times L = \{(x^+, x^-)\}$ defines a relation on $(\mathbb{R}^{2m}, \omega)$. If we want to describe this relation by a generating function of the midpoints, we endow the product space with the symplectic form $\delta_{1/2}\omega$ which makes $L \times L$ a canonical relation on $(\mathbb{R}^{2m}, \omega)$. However, if we want to "describe" the relation $\{(x^+, x^-)\}$ by another λ-point $x = \lambda x^+ + (1-\lambda)x^-$ on the chord joining the pair (x^+, x^-), this relation cannot be canonical anymore. In other words, if we want to describe the relation by a *generating function* of the λ-points, we must *weight differently* the symplectic form ω on each copy of $\mathbb{R}^{2m}$ in such a way as to account for the fact that we are describing the relation on $(\mathbb{R}^{2m}, \omega)$ in an *asymmetrical* way. The weights given in formula (4.1) for $\delta_\lambda\omega$ correctly account for this asymmetry.

Therefore, for each $\lambda \in \mathbb{R} \setminus \{0, 1\}$, a Lagrangian submanifold $\Lambda \subset (\mathbb{R}^{2m} \times \mathbb{R}^{2m}, \delta_\lambda\omega)$ can be seen as defining a λ-*weighted symplectic relation* on $(\mathbb{R}^{2m}, \omega)$. Hence, in particular, $\mathcal{L}_\lambda = \Phi_\lambda(L \times L) \subset (T\mathbb{R}^{2m}, \dot{\omega})$ defines a λ-weighted symplectic relation on $(\mathbb{R}^{2m}, \omega)$ which can locally be described by a λ-weighted generating function of the λ-points, whenever $\mathcal{L}_\lambda$ locally projects regularly to the zero section of $(T\mathbb{R}^{2m}, \dot{\omega})$.

Thus, $\mathcal{L}_\lambda$ is a Lagrangian submanifold of the $4m$-dimensional symplectic tangent bundle. The restriction of the projection of a Lagrangian bundle to a Lagrangian submanifold of the total space of this bundle is called a **Lagrangian map** [2]. So we obtain the result:

Proposition 4.2. *$pr|_{\mathcal{L}_\lambda} : \mathcal{L}_\lambda \to \mathbb{R}^{2m}$ is a Lagrangian map.*

The set of critical values of a Lagrangian map is called a **caustic** and from Theorem 3.2 we have

Corollary 4.3. *The caustic of $pr|_{\mathcal{L}_\lambda}$ is $E_\lambda(L)$.*

Definition 4.4. $E_\lambda(L)$ and $E_\lambda(\widetilde{L})$ are **Lagrangian equivalent** if the Lagrangian maps $pr|_{\mathcal{L}_\lambda}$ and $pr|_{\widetilde{\mathcal{L}}_\lambda}$ are Lagrangian equivalent (see [2]).

It follows from above definitions:

Proposition 4.5. *The classification of $E_\lambda(L)$ by Lagrangian equivalence is affine symplectic invariant, i.e., invariant under the standard action of the affine symplectic group on $(\mathbb{R}^{2m}, \omega)$.*

Definition 4.6. From the above proposition, we also use the terms **affine-Lagrangian equivalence** and **affine-Lagrangian stability** for Lagrangian equivalence and Lagrangian stability (see [2]) of an affine equidistant E_λ of a Lagrangian submanifold $L \subset (\mathbb{R}^{2m}, \omega)$.

We now rely on the well known fact that any smooth Lagrangian submanifold L of a symplectic affine space can be locally described as the graph of the differential of a certain generating function.

Thus, let L^+ and L^- denote germs of L at the points a^+ and a^-.

Proposition 4.7. *If the pair a^+, a^- is k-parallel ($k = 1, \cdots, m$) then there exists canonical coordinates (p, q) on $\mathbb{R}^{2m}$ and function germs S^+ and S^- such that*

$$L^+ : p_i = \frac{\partial S^+}{\partial q_i}(q_1, \cdots, q_m), \text{ for } i = 1, \cdots, m \tag{4.4}$$

$$L^- : \begin{cases} p_j = \ \ \frac{\partial S^-}{\partial q_j}(q_1, \cdots, q_k, p_{k+1}, \cdots, p_m), \text{ for } j = 1, \cdots, k, \\ q_l = -\frac{\partial S^-}{\partial p_l}(q_1, \cdots, q_k, p_{k+1}, \cdots, p_m), \text{ for } l = k+1, \cdots, m \end{cases} \tag{4.5}$$

and $d^2S^+(q^+_{a,1}, \cdots, q^+_{a,m}) = 0$ and $d^2S^-(p^-_{a,1}, \cdots, p^-_{a,k}, q^-_{a,k+1}, \cdots, p^-_{a,m}) = 0$, where $a^+ = (p^+_a, q^+_a)$ and $a^- = (p^-_a, q^-_a)$.

Proof. We can find a linear symplectic change of coordinates such that the tangent (affine) spaces have the following form $T_{a^+}L^+ = \{p = p^+_a\}$, where $a^+ = (p^+_a, q^+_a)$ and $T_{a^-}L^- = \{p_1 = p^-_{a,1}, \cdots, p_k = p^-_{a,k}, q_{k+1} = q^-_{a,k+1}, \cdots, q_m = q^-_{a,m}\}$, where $a^- = (p^-_a, q^-_a)$. Since L is a smooth Lagrangian submanifold, it follows from standard considerations that it can be described locally by differentials of generating functions of the forms stated above in neighborhoods of a^+ and a^-, in which case we have that $d^2S^+|a^+ = d^2S^-|a^- = 0$. $\square$

From the above, we state the main result of this section, which shall be used in all that follows.

Let the arguments of the function S^+ be denoted by $(q^+_1, \cdots, q^+_m)$ and the arguments of the function S^- by $(q^-_1, \cdots, q^-_k, p^-_{k+1}, \cdots, p^-_m)$. Let $q = (q_1, \cdots, q_m)$, $p = (p_1, \cdots, p_m)$, $\dot{q} = (\dot{q}_1, \cdots, \dot{q}_m)$, $\dot{p} = (\dot{p}_1, \cdots, \dot{p}_m)$.

Also, let $\beta = (\beta_1, \cdots, \beta_m)$ and, for any $k < m$, let $[k] = \{1, \cdots, k\}$, so that $\beta_{[k]} = (\beta_1, \cdots, \beta_k)$, and $\alpha_{[m]\setminus[k]} = (\alpha_{k+1}, \cdots, \alpha_m)$.

Let $L^+ \times L^-$ denote the germ of $L \times L$ at the point $(a^+, a^-) \in L \times L$ so that $\mathcal{L}_\lambda = \Phi_\lambda(L^+ \times L^-)$ is the germ at $(a, \dot{a})$, where $a = \lambda a^+ + (1-\lambda)a^-$, $\dot{a} = \lambda a^+ - (1-\lambda)a^-$, of a smooth Lagrangian submanifold of the $4m$-dimensional symplectic tangent bundle $(T\mathbb{R}^{2m}, \dot{\omega})$.

The restriction to $\mathcal{L}_\lambda$ of the projection $pr : T\mathbb{R}^{2m} \to \mathbb{R}^{2m}$ defines a germ of Lagrangian map. We say that a function-germ

$$G : \mathbb{R}^{2m} \times \mathbb{R}^s \ni (p, q, \kappa) \mapsto G(p, q, \kappa) \in \mathbb{R}$$

is a **generating family** of $\mathcal{L}_\lambda$ if

$$\mathcal{L}_\lambda = \left\{ (\dot{p}, \dot{q}, p, q) : \exists \kappa \in \mathbb{R}^s \ \dot{p} = \frac{\partial G}{\partial q},\ \dot{q} = -\frac{\partial G}{\partial p},\ \frac{\partial G}{\partial \kappa} = 0 \right\}.$$

We have the following result:

Theorem 4.8. *If the pair a^+, a^- is k-parallel and L^+ and L^- are given by (4.4) and (4.5) then the germ of $\mathcal{L}_\lambda$ at $(a, \dot{a})$ is generated by the germ of the generating family F_λ which is given by*

$$F_\lambda(p, q, \alpha_{[m]\setminus[k]}, \beta) = \tag{4.6}$$
$$2\lambda^2 S^+\left(\tfrac{q+\beta}{2\lambda}\right) - 2(1-\lambda)^2 S^-\left(\tfrac{q_{[k]}-\beta_{[k]}\ ,\ p_{[m]\setminus[k]}-\alpha_{[m]\setminus[k]}}{2(1-\lambda)}\right)$$
$$- \textstyle\sum_{i=1}^k p_i\beta_i + \frac{1}{2}\sum_{j=k+1}^m q_j\alpha_j - p_j\beta_j - \alpha_j\beta_j - p_j q_j.$$

Proof. We show that

$$\mathcal{L}_\lambda = \left\{ (\dot{p}, \dot{q}, p, q) : \exists (\alpha, \beta)\ \dot{p} = \frac{\partial F_\lambda}{\partial q},\ \dot{q} = -\frac{\partial F_\lambda}{\partial p},\ \frac{\partial F_\lambda}{\partial \alpha} = \frac{\partial F_\lambda}{\partial \beta} = 0 \right\}.$$

We have for $i = 1, \cdots, k$ and $j = k+1, \cdots, m$

$$\dot{p}_i = \lambda \frac{\partial S^+}{\partial q_i^+}\left(\frac{q+\beta}{2\lambda}\right) - (1-\lambda)\frac{\partial S^-}{\partial q_i^-}\left(\frac{q_{[k]}-\beta_{[k]}, p_{[m]\setminus[k]}-\alpha_{[m]\setminus[k]}}{2(1-\lambda)}\right), \tag{4.7}$$

$$\dot{p}_j = \lambda \frac{\partial S^+}{\partial q_j^+}\left(\frac{q+\beta}{2\lambda}\right) + \frac{1}{2}(\alpha_j - p_j), \tag{4.8}$$

$$\dot{q}_i = \beta_i, \tag{4.9}$$

$$\dot{q}_j = (1-\lambda)\frac{\partial S^-}{\partial p_j^-}\left(\frac{q_{[k]}-\beta_{[k]}, p_{[m]\setminus[k]}-\alpha_{[m]\setminus[k]}}{2(1-\lambda)}\right) + \frac{1}{2}(\beta_j + q_j), \tag{4.10}$$

$$\frac{\partial F_\lambda}{\partial \alpha_j} = (1-\lambda)\frac{\partial S^-}{\partial p_j^-}\left(\frac{q_{[k]}-\beta_{[k]}, p_{[m]\setminus[k]}-\alpha_{[m]\setminus[k]}}{2(1-\lambda)}\right) + \frac{1}{2}(q_j - \beta_j) = 0, \tag{4.11}$$

$$\frac{\partial F_\lambda}{\partial \beta_i} = \lambda \frac{\partial S^+}{\partial q_i^+}\left(\frac{q+\beta}{2\lambda}\right) + (1-\lambda)\frac{\partial S^-}{\partial q_i^-}\left(\frac{q_{[k]}-\beta_{[k]}, p_{[m]\setminus[k]}-\alpha_{[m]\setminus[k]}}{2(1-\lambda)}\right) - p_i = 0, \tag{4.12}$$

$$\frac{\partial F_\lambda}{\partial \beta_j} = \lambda \frac{\partial S^+}{\partial q_j^+}\left(\frac{q+\beta}{2\lambda}\right) - \frac{1}{2}(\alpha_j + p_j) = 0. \tag{4.13}$$

By (4.9) we get $\beta_i = \dot{q}_i$ for $i = 1, \cdots, k$. (4.10) and (4.11) imply that $\beta_j = \dot{q}_j$ for $j = k+1, \cdots, m$. By (4.8) and (4.13) we have $\alpha_j = \dot{p}_j$ for $j = k+1, \cdots, m$. Thus we eliminate $(\alpha_{[m]\setminus[k]}, \beta)$.

Then (4.7) implies that for $i = 1, \cdots, k$

$$\dot{p}_i = \lambda \frac{\partial S^+}{\partial q_i^+}\left(\frac{q+\dot{q}}{2\lambda}\right) - (1-\lambda)\frac{\partial S^-}{\partial q_i^-}\left(\frac{q_{[k]} - \dot{q}_{[k]}, p_{[m]\setminus[k]} - \dot{p}_{[m]\setminus[k]}}{2(1-\lambda)}\right). \tag{4.14}$$

(4.12) implies that for $i = 1, \cdots, k$

$$p_i = \lambda \frac{\partial S^+}{\partial q_i^+}\left(\frac{q+\dot{q}}{2\lambda}\right) + (1-\lambda)\frac{\partial S^-}{\partial q_i^-}\left(\frac{q_{[k]} - \dot{q}_{[k]}, p_{[m]\setminus[k]} - \dot{p}_{[m]\setminus[k]}}{2(1-\lambda)}\right). \tag{4.15}$$

By (4.8) and (4.13) we have for $j = k+1, \cdots, m$

$$\frac{1}{2\lambda}(\dot{p}_j + p_j) = \frac{\partial S^+}{\partial q_j^+}\left(\frac{q+\dot{q}}{2\lambda}\right). \tag{4.16}$$

(4.10) and (4.11) imply that for $j = k+1, \cdots, m$

$$\frac{1}{2(1-\lambda)}(q_j - \dot{q}_j) = -(1-\lambda)\frac{\partial S^-}{\partial p_j^-}\left(\frac{q_{[k]} - \dot{q}_{[k]}, p_{[m]\setminus[k]} - \dot{p}_{[m]\setminus[k]}}{2(1-\lambda)}\right). \tag{4.17}$$

If (p^+, q^-) (p^-, q^-) are points in L^+ and L^- described by (4.4) and (4.5) respectively then (4.14)-(4.17) describe $\mathcal{L}_\lambda$ in coordinates given by (3.1)-(3.2). $\square$

Remark 4.9. It is clear from the form of the generating family, given by (4.6), that *the degree of parallelism is the corank of the singularity* i. e. the corank of the Hessian of the function

$$\mathbb{R}^{2m-k} \ni (\alpha_{[m]\setminus[k]}, \beta) \mapsto F_\lambda(p_a, q_a, \alpha_{[m]\setminus[k]}, \beta) \in \mathbb{R}$$

Theorem 4.10 ([2])**.** *Two germs of Lagrangian maps are Lagrangian equivalent if and only if the germs of their generating families are stably $\mathcal{R}^+$-equivalent.*

Corollary 4.11. *Let L and $\tilde{L}$ be smooth closed Lagrangian submanifolds of the symplectic affine space $(\mathbb{R}^{2m}, \omega)$. Germs $E_\lambda(L)$ and $E_\lambda(\tilde{L})$ are Lagrangian equivalent if and only if the corresponding germs of generating families for $\mathcal{L}_\lambda$ and $\tilde{\mathcal{L}}_\lambda$ are stably $\mathcal{R}^+$-equivalent.*

5. Singularities of equidistants of Lagrangian submanifolds

In this section we study the singularities of momentary equidistants of closed Lagrangian submanifolds up to Lagrangian equivalence. Remind that, for $E_\lambda(L)$, Lagrangian stability is affine-Lagrangian stability (Proposition 4.5 and Definition 4.6). We have the following results:

Theorem 5.1. *Any caustic of stable Lagrangian singularity in the $4m$-dimensional symplectic tangent bundle $(T\mathbb{R}^{2m}, \dot\omega)$ is realizable as $E_\lambda(L)$, for some smooth closed Lagrangian submanifold L in $(\mathbb{R}^{2m}, \omega)$.*

Corollary 5.2. *For a smooth Lagrangian curve L, generic singularities of $E_\lambda(L)$ are cusps. In the neighborhood of its regular points, $E_\lambda(L)$ is a smooth curve in $(\mathbb{R}^2, \omega)$.*

Corollary 5.3. *For a smooth Lagrangian surface L, generic singularities of $E_\lambda(L)$ can be cusps A_3, swallowtails A_4, butterflies A_5, hyperbolic umbilics D_4^+, elliptic umbilics D_4^-, or parabolic umbilics D_5. In the neighborhood of its regular points, $E_\lambda(L)$ is a 3-dimensional smooth submanifold of $(\mathbb{R}^4, \omega)$.*

Proof of Theorem 5.1. We use the method described in [2]. For a fixed λ, let $x = (p, q)$ and $\kappa = (\alpha, \beta)$. From (4.6) we easily see that

$$\operatorname{rank}_{(a,\dot a)} \left[\frac{\partial^2 F_\lambda}{\partial \kappa^2}, \ \frac{\partial^2 F_\lambda}{\partial \kappa \partial x} \right] = 2m - k,$$

hence is equal to the dimension of κ-space. Now we must find such S^+ and S^- that $F_\lambda(x, \kappa)$ is a $\mathcal{R}^+$-versal deformation of $A - D - E$ singularities.

By Proposition 4.7 we obtain that

$$S^+(q^+) = \sum_{i=1}^{m} p^+_{a,i}(q^+_i - q^+_{a,i}) + S^+_3(q^+ - q^+_a)$$

$$\begin{aligned} S^-(q^-_{[k]}, p^-_{[m]\setminus[k]}) \ &= \ \sum_{i=1}^{k} p^-_{a,i}(q^-_i - q^-_{a,i}) - \sum_{i=k+1}^{m} q^-_{a,i}(p^-_i - p^-_{a,i}) + \\ &+ \ S^-_3(q^-_{[k]} - q^-_{a,[k]}, p^-_{[m]\setminus[k]} - p^-_{a,[m]\setminus[k]}), \end{aligned}$$

where$S_3^\pm \in \mathfrak{m}^3$ ($\mathfrak{m}$ is the maximal ideal of the ring of smooth function-germs on $\mathbb{R}^n$ at 0).

We write the generating families in coordinates $\tilde p = p - p_a$, $\tilde q = q - q_a$, $s = \alpha - \dot p_a$, $t = \beta - \dot q_a$, where $a = (p_a, q_a)$, $\dot a = (\dot p_a, \dot q_a)$. Then by

Theorem 4.8 we obtain

$$F_\lambda(\tilde p,\tilde q,s,t)= \tag{5.1}$$
$$2\lambda^2 S_3^+\left(\tfrac{\tilde q+t}{2\lambda}\right)-2(1-\lambda)^2 S_3^-\left(\tfrac{\tilde q_{[k]}-t_{[k]}\ ,\ \tilde p_{[m]\setminus[k]}-s_{[m]\setminus[k]}}{2(1-\lambda)}\right)$$
$$-\textstyle\sum_{i=1}^k \tilde p_i t_i+\frac12\sum_{j=k+1}^m \tilde q_j s_j-\tilde p_j t_j-s_j t_j-\tilde p_j\tilde q_j+\sum_{l=1}^m \dot p_{a,l}\tilde q_l-\dot q_{a,l}\tilde p_l$$

$$f_\lambda(s,t)=F_\lambda(0,0,s,t)= \tag{5.2}$$
$$2\lambda^2 S_3^+\left(\tfrac{t}{2\lambda}\right)-2(1-\lambda)^2 S_3^-\left(\tfrac{-t_{[k]},-s_{[m]\setminus[k]}}{2(1-\lambda)}\right)-\tfrac12\textstyle\sum_{j=k+1}^m s_j t_j$$

In order to prove Theorem 5.1, we analyze separately the cases of 1-parallelism and 2-parallelism, in every dimension.

To show that A_k singularity can be realizable as $E_\lambda(L)$, for 1-parallelism and $k\le 2m+1$ we use the generating family of the form (5.1) for $k=1$. To realize A_{2l} singularity take the following function-germs

$$S_3^+(\tilde q^+)=\lambda(\tilde q_1^+)^3+(\tilde q_1^+)^{2l+1}+\sum_{i=2}^{l}\tilde q_i^+(\tilde q_1^+)^{2i-1},$$

$$S_3^-(\tilde q_1^-,\tilde p_2^-,\cdots,\tilde p_m^-)=-(1-\lambda)(\tilde q_1^-)^3+\sum_{i=2}^{l-1}\tilde p_i^-(\tilde q_1^-)^{2(l-i+1)}.$$

A_{2l+1} singularity is realizable by the following function-germs

$$S_3^+(\tilde q^+)=\lambda(\tilde q_1^+)^3+(\tilde q_1^+)^{2l+2}+\sum_{i=2}^{l}\tilde q_i^+(\tilde q_1^+)^{2i-1},$$

$$S_3^-(\tilde q_1^-,\tilde p_2^-,\cdots,\tilde p_m^-)=-(1-\lambda)(\tilde q_1^-)^3+\sum_{i=2}^{l}\tilde p_i^-(\tilde q_1^-)^{2(l-i+2)}.$$

Now to show that D_k $(k\ge 4)$ or E_k $(k=6,7,8)$ singularity can be realizable as $E_\lambda(L)$, for 2-parallelism and $k\le 2m+1$, we use the generating family of the form (5.1) with $k=2$. The following singularities are realizable by the following generating functions:

$$D_{2l}:$$

$$S_3^+(\tilde q^+)=\lambda(\tilde q_1^+)^3+\tilde q_2^+(\tilde q_1^+)^2\pm(\tilde q_2^+)^{2l-1}+\lambda(\tilde q_2^+)^3+\sum_{i=2}^{l-1}\tilde q_{i+1}^+(\tilde q_2^+)^{2i-1},$$

$$S_3^-(\tilde q_{[2]}^-,\tilde p_{[m]\setminus[2]}^-)=-(1-\lambda)(\tilde q_1^-)^3-(1-\lambda)(\tilde q_2^-)^3+\sum_{i=2}^{l-2}\tilde p_{i+1}^-(\tilde q_2^-)^{2(l-i)}.$$

$$D_{2l+1}:$$

$$S_3^+(\tilde{q}^+) = \lambda(\tilde{q}_1^+)^3 + \tilde{q}_2^+(\tilde{q}_1^+)^2 \pm (\tilde{q}_2^+)^{2l} + \lambda(\tilde{q}_2^+)^3 + \sum_{i=2}^{l-1} \tilde{q}_{i+1}^+(\tilde{q}_2^+)^{2i-1},$$

$$S_3^-(\tilde{q}_{[2]}^-, \tilde{p}_{[m]\setminus[2]}^-) = -(1-\lambda)(\tilde{q}_1^-)^3 - (1-\lambda)(\tilde{q}_2^-)^3 + \sum_{i=2}^{l-1} \tilde{p}_{i+1}^-(\tilde{q}_2^-)^{2(l-i+1)}.$$

$$E_6:$$
$$S_3^+(\tilde{q}^+) = (\tilde{q}_1^+)^3 \pm (\tilde{q}_2^+)^4 + \lambda\tilde{q}_1^+(\tilde{q}_2^+)^2 + \lambda(\tilde{q}_2^+)^3 + \tilde{q}_1^+(\tilde{q}_2^+)^2\tilde{q}_3^+,$$
$$S_3^-(\tilde{q}_{[2]}^-, \tilde{p}_{[m]\setminus[2]}^-) = -(1-\lambda)\tilde{q}_1^-(\tilde{q}_2^-)^2 - (1-\lambda)(\tilde{q}_2^-)^3.$$

$$E_7:$$
$$S_3^+(\tilde{q}^+) = (\tilde{q}_1^+)^3 + \tilde{q}_1^+(\tilde{q}_2^+)^2 + \lambda\tilde{q}_1^+(\tilde{q}_2^+)^2 + \lambda(\tilde{q}_2^+)^3 + (\tilde{q}_2^+)^3\tilde{q}_3^+,$$
$$S_3^-(\tilde{q}_{[2]}^-, \tilde{p}_{[m]\setminus[2]}^-) = -(1-\lambda)\tilde{q}_1^-(\tilde{q}_2^-)^2 - (1-\lambda)(\tilde{q}_2^-)^3 + (\tilde{q}_2^-)^4\tilde{p}_3^-.$$

$$E_8:$$
$$S_3^+(\tilde{q}^+) = (\tilde{q}_1^+)^3 + (\tilde{q}_2^+)^5 + \lambda\tilde{q}_1^+(\tilde{q}_2^+)^2 + \lambda(\tilde{q}_2^+)^3 + \tilde{q}_1^+(\tilde{q}_2^+)^2\tilde{q}_3^+ + \tilde{q}_1^+(\tilde{q}_2^+)^3\tilde{q}_4^+,$$
$$S_3^-(\tilde{q}_{[2]}^-, \tilde{p}_{[m]\setminus[2]}^-) = -(1-\lambda)\tilde{q}_1^-(\tilde{q}_2^-)^2 - (1-\lambda)(\tilde{q}_2^-)^3 + (\tilde{q}_2^-)^3\tilde{p}_3^-.$$

By long but straightforward calculations one can show that (5.1) is a $\mathcal{R}^+$-versal deformation of (5.2) for the above choices of $S_3^\pm$. ☐

6. The GCS of a Lagrangian submanifold: the criminant

We now begin the study of singularities of the Global Centre Symmetry set of a smooth closed Lagrangian submanifold $L \subset (\mathbb{R}^{2m}, \omega)$.

Remind that, in terms of the projection $\pi : \mathbb{R} \times \mathbb{R}^{2m} \ni (\lambda, x) \mapsto x \in \mathbb{R}^{2m}$, Definitions 2.6 and 2.7 states that $GCS(L)$ is the locus of critical points of $\pi|_{\mathbb{E}(L)}$. From Remarks 2.9 and 2.11, $GCS(L)$ consists of two parts which can be further refined to comprise three parts:

(i) the **Wigner caustic** $E_{1/2}(L)$.

(ii) the **centre symmetry caustic** $\Sigma'(L)$, consisting of the λ-family of π-projections of singularities of $\mathbb{E}(L)$, excluding the Wigner caustic.

(iii) the **criminant** $\Delta(L)$, being the π-projection of smooth parts of the extended wave front $\mathbb{E}(L)$ that are tangent to the fibers of π.

The classification of the Wigner caustic of a Lagrangian submanifold L has been mostly carried out in the last section, since the Wigner caustic is the $\lambda = 1/2$ affine equidistant. In a subsequent paper [5], we study $E_{1/2}(L)$ in a neighborhood L, considered in a broader sense, that is, considering pairs of points of the type $(a, a) \in L \times L$ as strongly parallel pairs. Then, in a neighborhood of L, we look for singularities

of the Wigner caustic that have maximal co-rank m. In terms of the generating families of section 4, these now have the special form

$$F_{1/2}(p,q,\beta) = \frac{1}{2}S(q+\beta) - \frac{1}{2}S(q-\beta) - \sum_{i=1}^{m} p_i\beta_i \ , \tag{6.1}$$

where S is the local generating function of a germ of the Lagrangian submanifold $L \subset (\mathbb{R}^{2m}, \omega)$. It follows immediately from (6.1) that only the generating families for singularities of co-rank m which are *odd* functions of β should be considered, in this case.

This point had already been made in [16], but, in order to study such singularities, we must consider classification *in the category of odd functions* [5]. It also implies $\mathbb{Z}_2$-symmetric singularities for the Wigner caustic on-shell. A first study of such symmetric singularities, for the case of surfaces in nonsymplectic $\mathbb{R}^4$, is presented in [13].

In order to study the centre symmetry caustic $\Sigma'(L)$ and the criminant $\Delta(L)$, the whole λ-family must be considered together. Due to the Lagrangian condition, we resort to a classification via generating families, as was done in sections 4 and 5 for the λ-equidistants.

From results of the previous sections we know that $E_\lambda(L)$ is the caustic of the Lagrangian submanifold $\mathcal{L}_\lambda = \Phi_\lambda(L \times L)$ in the Lagrangian fiber bundle $(T\mathbb{R}^{2m}, \dot{\omega}) \to \mathbb{R}^{2m}$, where Φ_λ be the λ-chord transformation given by equations (3.1) and (3.2). By Theorem 4.8 the generating family for $\mathcal{L}_\lambda$ is given by $F_\lambda(p,q,\alpha,\beta)$ of the form (4.6). Since $\mathbb{E}(L)$ is the union of $\{\lambda\} \times E_\lambda$ we obtain that the germ of $\mathbb{E}(L)$ is described in the following way (for $\kappa = (\alpha,\beta)$):

Proposition 6.1. $\mathbb{E}(L) = \left\{(\lambda,p,q) : \exists\kappa \ \frac{\partial F_\lambda}{\partial \kappa} = 0, \ \det\left[\frac{\partial^2 F_\lambda}{\partial\kappa_i\partial\kappa_j}\right] = 0\right\}$.

We now find a Lagrangian fiber bundle and the germ of a Lagrangian submanifold $\mathcal{L}$ in this bundle such that $\mathbb{E}(L)$ is the caustic of $\mathcal{L}$.

Let us consider the fiber bundle

$$Pr : T^*\mathbb{R} \times T\mathbb{R}^{2m} \ni ((\lambda^*,\lambda),(\dot{p},\dot{q},p,q)) \mapsto (\lambda,(p,q)) \in \mathbb{R}\times\mathbb{R}^m. \tag{6.2}$$

The above bundle with the canonical symplectic structure

$$d\lambda^* \wedge d\lambda + \dot{\omega}$$

is a Lagrangian fiber bundle. For F_λ given by (4.6) in Theorem 4.8, let

$$F(\lambda,p,q,\alpha,\beta) = F_\lambda(p,q,\alpha,\beta).$$

For $\kappa = (\alpha,\beta) = (\alpha_{[m]\setminus[k]},\beta) = (\kappa_1,\cdots,\kappa_{2m-k})$, we have the result:

Proposition 6.2. *The germ of $\mathbb{E}(L)$ is the caustic of the germ of a Lagrangian submanifold $\mathcal{L}$ of the Lagrangian fiber bundle $(T^*\mathbb{R} \times T\mathbb{R}^{2m}, d\lambda^* \wedge d\lambda + \dot{\omega})$ generated by the family F in the following way*
(6.3)
$$\mathcal{L} = \left\{((\lambda^*, \lambda), (\dot{p}, \dot{q}, p, q)) : \exists \kappa \;\; \lambda^* = \frac{\partial F}{\partial \lambda},\ \dot{p} = \frac{\partial F}{\partial q},\ \dot{q} = -\frac{\partial F}{\partial p},\ \frac{\partial F}{\partial \kappa} = 0\right\}.$$

6.1. **Geometric characterization of the criminant of the GCS of a Lagrangian submanifold.** Remind that the criminant $\Delta(L)$ is the (closure of) the image under π_r of the set of regular points of $\mathbb{E}(L)$ which are critical points of the projection π restricted to the regular part of $\mathbb{E}(L)$. That is, the criminant $\Delta(L)$ is the envelope of the family of regular parts of momentary equidistants. We find the condition for the tangency to the fibers of the projection $\pi : (\lambda, p, q) \mapsto (p, q)$.

Proposition 6.3. *If (λ, a) is a regular point of $\mathbb{E}(L)$ then there exists a 1-parallel pair a^+, a^- such that $a = \lambda a^+ + (1-\lambda)a^-$.*

Proof. (λ_a, p_a, q_a) is a regular point of $\mathbb{E}(L)$ then the rank of the map
$$\kappa \mapsto \left(\frac{\partial F}{\partial \kappa}(\lambda_a, p_a, q_a, \kappa),\ \det\left[\frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j}(\lambda_a, p_a, q_a, \kappa)\right]\right)$$
is maximal $2m-k$. It implies that corank $\left[\frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j}(\lambda_a, p_a, q_a, \kappa_a)\right]$ is 1. By Remark 4.9 we obtain that a^+, a^- is a 1-parallel pair. $\square$

Proposition 6.4. *Let $(\lambda_a, a) = (\lambda_a, p_a, q_a)$ be a regular point of $\mathbb{E}(L)$. Then the fiber of π_r is tangent to $\mathbb{E}(L)$ at (λ_a, p_a, q_a) if and only if*
$$\text{(6.4)} \qquad rank\left[\frac{\partial^2 F}{\partial \lambda \partial \kappa_j}, \frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j}\right] = rank\left[\frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j}\right] = 2m-2$$
at $(\lambda_a, p_a, q_a, \kappa_a)$ such that
$$\frac{\partial F}{\partial \kappa}(\lambda_a, p_a, q_a, \kappa_a) = 0,\ \det\left[\frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j}(\lambda_a, p_a, q_a, \kappa_a)\right] = 0.$$

Proof. By Proposition 6.3 if (λ_a, p_a, q_a) is a regular point of $\mathbb{E}(L)$ then the rank of the map
$$\kappa \mapsto \left(\frac{\partial F}{\partial \kappa}(\lambda_a, p_a, q_a, \kappa),\ \det\left[\frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j}(\lambda_a, p_a, q_a, \kappa)\right]\right)$$
is maximal $2m-1$. We also have that rank $\left[\frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j}(\lambda_a, p_a, q_a, \kappa_a)\right]$ is $2m-2$ which implies that one of the columns of this matrix is linearly

dependent on the others. For simplicity we assume that this is the first column. Thus a rank of the map

$$\kappa \mapsto \left(\frac{\partial F}{\partial \kappa_{[2m-1]\setminus[1]}}(\lambda_a, p_a, q_a, \kappa),\ \det \left[\frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j}(\lambda_a, p_a, q_a, \kappa) \right] \right)$$

is maximal $2m-1$. By the implicit function theorem there exists a smooth map germ $\mathcal{K} : \mathbb{R}^{2m+1}_e \to \mathbb{R}^{2m-1}$ at(λ_a, p_a, q_a), such that $\kappa = \mathcal{K}(\lambda, p, q)$ if and only if

$$\frac{\partial F}{\partial \kappa_{[2m-1]\setminus[1]}}(\lambda, p, q, \kappa) = 0,\ \det \left[\frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j}(\lambda, p, q, \kappa) \right] = 0.$$

Then the germ of $\mathbb{E}(L)$ at(λ_a, p_a, q_a) has the following form:

$$\mathbb{E}(L) = \left\{ (\lambda, p, q) : \frac{\partial F}{\partial \kappa_1}(\lambda, p, q, \mathcal{K}(\lambda, p, q)) = 0 \right\}.$$

The fiber of π_r is tangent to $\mathbb{E}(L)$ at (λ_a, p_a, q_a) if and only if

$$\frac{\partial}{\partial \lambda} \left(\frac{\partial F}{\partial \kappa_1}(\lambda, p, q, \mathcal{K}(\lambda, p, q)) \right) (\lambda_a, p_a, q_a) = 0,$$

which can be rewritten as
(6.5)
$$\frac{\partial^2 F}{\partial \lambda \partial \kappa_1}(\lambda_a, p_a, q_a, \kappa_a) + \sum_{j=1}^{2m-1} \frac{\partial^2 F}{\partial \kappa_j \partial \kappa_1}(\lambda_a, p_a, q_a, \kappa_a) \frac{\partial \mathcal{K}_j}{\partial \lambda}(\lambda_a, p_a, q_a) = 0.$$

On the other hand, differentiating $\frac{\partial F}{\partial \kappa_{[2m-1]\setminus[1]}}(\lambda, p, q, \mathcal{K}(\lambda, p, q)) = 0$ with respect to λ we obtain for $i = 2, \cdots, 2m-1$
(6.6)
$$\frac{\partial^2 F}{\partial \lambda \partial \kappa_i}(\lambda_a, p_a, q_a, \kappa_a) + \sum_{j=1}^{2m-1} \frac{\partial^2 F}{\partial \kappa_j \partial \kappa_i}(\lambda_a, p_a, q_a, \kappa_a) \frac{\partial \mathcal{K}_j}{\partial \lambda}(\lambda_a, p_a, q_a) = 0.$$

Thus (6.5)-(6.6) imply (6.4). But also (6.6) and (6.4) imply (6.5). □

Theorem 6.5. *The point $a = \lambda a^+ + (1-\lambda) a^-$ belongs to the criminant $\Delta(L)$ of the Global Centre Symmetry set of L if and only if there exists a bitangent hyperplane to L at points a^+ and a^-.*

Proof. First assume that (λ, a) is a regular point of $\mathbb{E}(L)$. By Propositions 6.3-6.4 a^+, a^- is a 1-parallel pair and $a = (p, q)$ is in the criminant

if and only if (λ, a) satisfies (6.4). Thus $\left[\frac{\partial^2 F}{\partial\kappa_i\partial\kappa_j}\right]$ has the following form

$$\frac{1}{2}\begin{bmatrix} \frac{\partial^2 S^+}{(\partial q_1^+)^2} - \frac{\partial^2 S^-}{(\partial q_1^-)^2} & \frac{\partial^2 S^+}{\partial q_1^+\partial q_2^+} & \cdots & \frac{\partial^2 S^+}{\partial q_1^+\partial q_m^+} & -\frac{\partial^2 S^-}{\partial q_1^-\partial p_2^-} & \cdots & -\frac{\partial^2 S^-}{\partial q_1^-\partial p_m^-} \\ \frac{\partial^2 S^+}{\partial q_1^+\partial q_2^+} & \frac{\partial^2 S^+}{(\partial q_2^+)^2} & \cdots & \frac{\partial^2 S^+}{\partial q_2^+\partial q_m^+} & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 S^+}{\partial q_1^+\partial q_m^+} & \frac{\partial^2 S^+}{\partial q_2^+\partial q_m^+} & \cdots & \frac{\partial^2 S^+}{(\partial q_m^+)^2} & 0 & \cdots & -1 \\ -\frac{\partial^2 S^-}{\partial q_1^-\partial p_2^-} & -1 & \cdots & 0 & -\frac{\partial^2 S^-}{(\partial p_2^-)^2} & \cdots & \frac{\partial^2 S^-}{\partial p_2^-\partial p_m^-} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial^2 S^-}{\partial q_1^-\partial p_m^-} & 0 & \cdots & -1 & \frac{\partial^2 S^-}{\partial p_2^-\partial p_m^-} & \cdots & -\frac{\partial^2 S^-}{(\partial p_m^-)^2} \end{bmatrix}$$

On the other hand

$$\frac{\partial^2 F}{\partial\lambda\partial\beta_1} = p_1^+ - p_1^- - \sum_{j=1}^{n} q_j^+ \frac{\partial^2 S^+}{\partial q_1^+\partial q_j^+} + q_1^- \frac{\partial^2 S^-}{(\partial q_1^-)^2} + \sum_{j=2}^{n} p_j^- \frac{\partial^2 S^-}{\partial q_1^-\partial p_j^-},$$

$$\frac{\partial^2 F}{\partial\lambda\partial\beta_i} = p_i^+ - \sum_{i=1}^{n} q_j^+ \frac{\partial^2 S^+}{\partial q_i^+\partial q_j^+}, \text{ for } i = 2, \cdots, m,$$

$$\frac{\partial^2 F}{\partial\lambda\partial\alpha_i} = q_i^- + q_1^- \frac{\partial^2 S^+}{\partial p_i^-\partial q_1^-} + \sum_{j=2}^{n} p_j^- \frac{\partial^2 S^+}{\partial p_i^-\partial p_j^-}, \text{ for } i = 2, \cdots, m,$$

where $q^+ = \frac{q+\beta}{2\lambda}$,$p^+ = \frac{\partial S^+}{\partial q^+}$ are coordinates of $a^+ \in L^+$ and $q_1^- = \frac{q_1-\beta_1}{2(1-\lambda)}$, $p^-_{[m]\setminus[2]} = \frac{p_{[m]\setminus[2]}-\alpha_{[m]\setminus[2]}}{2(1-\lambda)}$, $p_1^- = \frac{\partial S^-}{\partial q_1^-}$, $q^-_{[m]\setminus[2]} = -\frac{\partial S^-}{\partial p^-_{[m]\setminus[2]}}$ are coordinates of $a^- \in L^-$.

Then (6.4) is equivalent to

$$(a^+ - a^-) \in T_{a^+}L^+ + T_{a^-}L^-, \tag{6.7}$$

since $T_{a^+}L^+$ is spanned by vectors $\sum_{j=1}^m \frac{\partial^2 S^+}{\partial q_i^+\partial q_j^+}\frac{\partial}{\partial p_j} + \frac{\partial}{\partial q_i}$ for $i = 1, \cdots, m$ and $T_{a^-}L^-$ is spanned by vectors $\frac{\partial^2 S^-}{(\partial q_1^-)^2}\frac{\partial}{\partial p_1} - \sum_{j=2}^m \frac{\partial^2 S^-}{\partial q_1^-\partial p_j^-}\frac{\partial}{\partial q_j} + \frac{\partial}{\partial q_1}$ and $\frac{\partial^2 S^-}{\partial p_i^-\partial q_1^-}\frac{\partial}{\partial p_1} - \sum_{j=2}^m \frac{\partial^2 S^-}{\partial p_i^-\partial p_j^-}\frac{\partial}{\partial q_j} + \frac{\partial}{\partial p_i}$ for $i = 2, \cdots, m$.

a^+, a^- is 1-parallel then (6.7) exactly means that there exists a bitangent hyperplane to L^+ at a^+ and to L^- at a^-. By continuity, a point in the closure of the set of points which satisfy (6.7) also satisfies this condition. □

Corollary 6.6. *If, for some λ, the point $a = \lambda a^+ + (1-\lambda)a^-$ belongs to the criminant $\Delta(L) \subset GCS(L)$, then the whole chord $l(a^+, a^-)$ belongs to $GCS(L)$. Equivalently, if there exists a bitangent hyperplane to L at points a^+ and a^-, then the chord $l(a^+, a^-)$ belongs to $GCS(L)$.*

In view of these results, we now generalize the notion of convexity of a curve on the plane.

Definition 6.7. A smooth closed Lagrangian submanifold L of the affine symplectic space $(\mathbb{R}^{2m}, \omega)$ is **weakly convex** if there is no bi-tangent hyperplane to L.

Corollary 6.8. *If L is a weakly convex closed Lagrangian submanifold of $(\mathbb{R}^{2m}, \omega)$ then the criminant $\Delta(L)$ of $GCS(L)$ is empty.*

7. Affine-Lagrangian stable singularities of the GCS of Lagrangian submanifolds

We now turn to the definition of an equivalence relation to be used for the classification of the singularities of $GCS(L)$. Due to the Lagrangian condition, we look for an equivalence of generating families.

Remind that, for the classification of $\mathbb{E}(\lambda)$ and $GCS(L)$, because λ is no longer fixed it has become an extra parameter that unfolds the generating families F. The naive approach is to consider the extended parameter space $\mathbb{R} \times \mathbb{R}^{2m} \ni (\lambda, x)$ for unfolding the generating families $f(\lambda, \kappa) = f_\lambda(\kappa)$ and classify their stable unfoldings in the usual way.

However, such a classification of $GCS(L)$ would not take into account the projection $\pi : \mathbb{R} \times \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ in a proper way, because it is not possible to introduce the notion of affine symplectic invariance for such a classification of $GCS(L)$, since the naive Lagrangian equivalence of $\mathbb{E}(L)$, as above, does not distinguish the affine time $\lambda \in \mathbb{R}$ from $x \in \mathbb{R}^{2m}$. Now, if $\mathcal{A} = (A, a)$ is an element of the affine symplectic group $iSp_{\mathbb{R}}^{2m} = Sp(2m, \mathbb{R}) \ltimes \mathbb{R}^{2m}$, with $A \in Sp(2m, \mathbb{R})$, $a \in \mathbb{R}^{2m}$, then

$$\mathcal{A} : (\mathbb{R}^{2m}, \omega) \supset L \to L' \subset (\mathbb{R}^{2m}, \omega) \ , \ x \mapsto \mathcal{A}x = Ax + a \ . \tag{7.1}$$

From this, we define the natural action

$$id_{T^*\mathbb{R}} \times \mathcal{A} \times \mathcal{A} : T^*\mathbb{R} \times \mathbb{R}^{2m} \times \mathbb{R}^{2m} \to T^*\mathbb{R} \times \mathbb{R}^{2m} \times \mathbb{R}^{2m} \ ,$$

$$(\lambda, \lambda^*, x^+, x^-) \mapsto (\lambda, \lambda^*, \mathcal{A}x^+, \mathcal{A}x^-) \ ,$$

which, via the chord transformation Φ_λ, induces an action

$$iSp_{\mathbb{R}}^{2m} \ \ni \ id_{T^*\mathbb{R}} \times \mathcal{A}_\Phi \ : \ T^*\mathbb{R} \times T\mathbb{R}^{2m} \supset \mathcal{L} \ \to \ \mathcal{L}' \subset T^*\mathbb{R} \times T\mathbb{R}^{2m},$$

$$id_{T^*\mathbb{R}} \times \mathcal{A}_\Phi : (\lambda, \lambda^*, \Phi_\lambda(x^+, x^-)) \mapsto (\lambda, \lambda^*, \Phi_\lambda(\mathcal{A}x^+, \mathcal{A}x^-)),$$

$$id_{T^*\mathbb{R}} \times \mathcal{A}_\Phi : (\lambda, \lambda^*, x, \dot{x}) \mapsto (\lambda, \lambda^*, Ax + a, A\dot{x} + (2\lambda - 1)a), \tag{7.2}$$

that commutes with projection $id_{T^*\mathbb{R}} \times pr : T^*\mathbb{R} \times T\mathbb{R}^{2m} \to T^*\mathbb{R} \times \mathbb{R}^{2m}$, that is, defining the obvious action $id_{\mathbb{R}} \times \mathcal{A}$ on $\mathbb{R} \times \mathbb{R}^{2m}$, we have

$$(id_{\mathbb{R}} \times \mathcal{A}) \circ (id_{T^*\mathbb{R}} \times pr) = (id_{T^*\mathbb{R}} \times pr) \circ (id_{T^*\mathbb{R}} \times \mathcal{A}_\Phi). \tag{7.3}$$

From the above and Proposition 6.2, we define a modified Lagrangian equivalence which takes into account projection $\pi : \mathbb{R} \times \mathbb{R}^{2m} \to \mathbb{R}^{2m}$.

Definition 7.1. Germs of Lagrangian submanifolds $\mathcal{L}$, $\widetilde{\mathcal{L}}$ of the fiber bundle $(T^*\mathbb{R} \times T\mathbb{R}^{2m}, d\lambda^* \wedge d\lambda + \dot{\omega})$ are **(1,2m)-Lagrangian equivalent** if there exists a symplectomorphism-germ Υ of $T^*\mathbb{R} \times T\mathbb{R}^{2m}$ such that $\Upsilon(\mathcal{L}) = \widetilde{\mathcal{L}}$ and the following diagram commutes:

$$\begin{array}{ccccccc} \mathcal{L} & \hookrightarrow & T^*\mathbb{R} \times T\mathbb{R}^{2m} & \xrightarrow{Pr} & \mathbb{R} \times \mathbb{R}^{2m} & \xrightarrow{\pi} & \mathbb{R}^{2m} \\ & & \downarrow \Upsilon & & \downarrow & & \downarrow \\ \widetilde{\mathcal{L}} & \hookrightarrow & T^*\mathbb{R} \times T\mathbb{R}^{2m} & \xrightarrow{Pr} & \mathbb{R} \times \mathbb{R}^{2m} & \xrightarrow{\pi} & \mathbb{R}^{2m} \end{array}$$

The first two vertical diffeomorphism-germs (from right to left) read:

$$x \mapsto X(x)\ ,\ (\lambda, x) \mapsto (\Lambda(\lambda, x), X(x)).$$

Moreover, germs $\mathcal{L}$, $\widetilde{\mathcal{L}}$ at $(\frac{1}{2}, a, \dot{a})$ are **(1,2m)-Lagrangian equivalent for** $\lambda = \frac{1}{2}$ if, in addition, for every $x \in \mathbb{R}^{2m}$

$$\Lambda(\frac{1}{2}, x) = \frac{1}{2}. \tag{7.4}$$

Remark 7.2. Condition (7.4) is introduced for the classification of the Wigner caustic $E_{\frac{1}{2}}(L)$ as a part of $GCS(L)$. If (7.4) is satisfied then the diffeomorphism (Λ, X) preserves the Wigner caustic.

Definition 7.3. $GCS(L)$ and $GCS(\widetilde{L})$ are **(1,2m)-Lagrangian equivalent** if $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ are (1,2m)-Lagrangian equivalent.

Remark 7.4. $(1, 2m)$-Lagrangian equivalence of germs of Lagrangian submanifolds of the Lagrangian fiber bundle $(T^*\mathbb{R} \times T\mathbb{R}^{2m}, d\lambda^* \wedge d\lambda + \dot{\omega})$ is the equivalence of bifurcations of Lagrangian maps (Section 10.1 in [2]), that is, diagrams of maps of the form:

$$D(\mathcal{L}) :\ \mathcal{L} \hookrightarrow T^*\mathbb{R} \times T\mathbb{R}^{2m} \xrightarrow{Pr} \mathbb{R} \times \mathbb{R}^{2m} \xrightarrow{\pi} \mathbb{R}^{2m}$$

Definition 7.5. A Lagrangian submanifold $\mathcal{L}$ is **(1,2m)-Lagrangian stable** if the diagram of maps $D(\mathcal{L})$ is stable, i.e. every Lagrangian submanifold $\widetilde{\mathcal{L}}$ with nearby diagram $D(\widetilde{\mathcal{L}})$ is $(1, 2m)$-Lagrangian equivalent to $\mathcal{L}$. $GCS(L)$ is **(1,2m)-Lagrangian stable** if $\mathcal{L}$ is (1,2m)-Lagrangian stable. In view of the following remark, we also use the term **affine-Lagrangian stability** for $(1, 2m)$-Lagrangian stability of $\mathcal{L}$ and $GCS(L)$.

Remark 7.6. The classification of $GCS(L)$ by $(1,2m)$-Lagrangian equivalence of $\mathcal{L}$ is *affine symplectic invariant* because, $\forall \mathcal{A} \in iSp^{2m}_{\mathbb{R}}$, the following diagram commutes (see (7.3)):

$$\begin{array}{ccccc}
T^*\mathbb{R} \times T\mathbb{R}^{2m} & \stackrel{Pr}{\longrightarrow} & \mathbb{R} \times \mathbb{R}^{2m} & \stackrel{\pi}{\rightarrow} & \mathbb{R}^{2m} \\
\downarrow id_{T^*\mathbb{R}} \times \mathcal{A}_\Phi & & \downarrow id_{\mathbb{R}} \times \mathcal{A} & & \downarrow \mathcal{A} \\
T^*\mathbb{R} \times T\mathbb{R}^{2m} & \stackrel{Pr}{\longrightarrow} & \mathbb{R} \times \mathbb{R}^{2m} & \stackrel{\pi}{\rightarrow} & \mathbb{R}^{2m}
\end{array}$$

for $\mathcal{A}$ and $\mathcal{A}_\Phi$ given by (7.1) and (7.2), so that, if $L' = \mathcal{A}(L)$, then $\mathcal{L}' = (id_{T^*\mathbb{R}} \times \mathcal{A}_\Phi)(\mathcal{L})$ and $\mathbb{E}(L) = (id_{\mathbb{R}} \times \mathcal{A})\mathbb{E}(L)$, $GCS(L') = \mathcal{A}(GCS(L))$.

Thus, for an affine symplectic invariant classification of $GCS(L)$, the generating families for $\mathcal{L}$ cannot be unfolded by the parameters $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^{2m}$ as if they were on an equal footing. However, in a natural way, the $(1,2m)$-Lagrangian equivalence of Lagrangian submanifolds of $T^*\mathbb{R} \times T\mathbb{R}^{2m}$ leads to the following equivalence of generating families.

Definition 7.7. The function-germs $F, \widetilde{F} : \mathbb{R} \times \mathbb{R}^{2m} \times \mathbb{R}^k \to \mathbb{R}$ are **(1,2m)-$\mathcal{R}^+$-equivalent** if there exists a diffeomorphism-germ

$$(\lambda, x, \kappa) \mapsto (\Lambda(\lambda, x), X(x), K(\lambda, x, \kappa))$$

and a smooth function-germ $g : \mathbb{R} \times \mathbb{R}^{2m} \to \mathbb{R}$ such that

$$\widetilde{F}(\lambda, x, \kappa) = F(\Lambda(\lambda, x), X(x), K(\lambda, x, \kappa)) + g(\lambda, x).$$

Germs F and $\widetilde{F}$ with the common (λ, x)-space $\mathbb{R} \times \mathbb{R}^{2m}$ of parameters, but in general, with their spaces of arguments of different dimensions are **stably (1,2m)-$\mathcal{R}^+$-equivalent** if there are nondegenerate quadratic forms Q in new arguments ξ and $\widetilde{Q}$ in new arguments $\tilde{\xi}$ such that $F + Q$ and $\widetilde{F} + \widetilde{Q}$ are $(1,2m)$-$\mathcal{R}^+$-equivalent. The germ F at $(\frac{1}{2}, a, \kappa_a)$ and the germ $\widetilde{F}$ at $(\frac{1}{2}, a, \tilde{\kappa}_a)$ are **(stably) (1,2m)-$\mathcal{R}^+$-equivalent for** $\lambda = \frac{1}{2}$ if, in addition, for every $x \in \mathbb{R}^m$ condition (7.4) is satisfied (the role of condition (7.4) is explained in Remark 7.2.

Remark 7.8. $(1,2m)$-$\mathcal{R}^+$-equivalence is a special case of Wassermann's $(1,2m)$-equivalence studied in [19]. See also Section 10.1 in [2], where relations between (r,s)-classification of families of functions ([19]), classification of bifurcations of caustics ([1] and [20]) and classification of bifurcations of Lagrangian maps (see Remark 7.4) were discussed.

We have the following result, whose proof is a minor modification for $(1,2m)$-Lagrangian equivalence of the proof of Theorem 4.10 in [2].

Proposition 7.9. *Germs of Lagrangian submanifolds $\mathcal{L}$, $\widetilde{\mathcal{L}}$ of the Lagrangian fiber bundle $(T^*\mathbb{R}\times T\mathbb{R}^{2m}, d\lambda^*\wedge d\lambda+\dot{\omega})$ are $(1,2m)$-Lagrangian equivalent if and only if the corresponding germs of generating families F and $\widetilde{F}$ are stably $(1,2m)$-$\mathcal{R}^+$-equivalent.*

Definition 7.10. The function-germ F at z is **(1,2m)-$\mathcal{R}^+$-stable** if for any neighborhood U of z in $\mathbb{R}\times\mathbb{R}^{2m}\times\mathbb{R}^k$ and representative function F' of the germ F defined on U, there exists a neighborhood V of F' in $C^\infty(U,\mathbb{R})$ (with the weak C^∞-topology) such that for any function $G'\in V$ there exists a point $z'\in U$ such that the germ of G' at z' is $(1,2m)$-$\mathcal{R}^+$-equivalent to F.

Remark 7.11. Let F be a generating family of $\mathcal{L}$. F is $(1,2m)$-$\mathcal{R}^+$-stable iff the corresponding germs of $\mathcal{L}$ and $GCS(L)$ are $(1,2m)$-Lagrangian stable.

Definitions 7.1-7.10 are the ones we were looking for. The following theorems show that the only affine-Lagrangian stable singularities of GCS are singularities of the criminant, the smooth part of the Wigner caustic and the "tangent" union of them.

Theorem 7.12. *Let $\lambda_a\neq\frac{1}{2}$. If F is the germ at (λ_a, a, κ_a) of a $(1,2m)$-$\mathcal{R}^+$-stable unfolding of $f\in\mathfrak{m}^2$ then F is stably $(1,2m)$-$\mathcal{R}^+$-equivalent to the germ of the trivial unfolding (if f has A_1 singularity) or to one of the following germs at $(0,0,0)$ of unfoldings of $f(t)=t^3$*

$$A_2^{A_k^\pm}\ :\ F(\lambda,x,t)=t^3+t\left(\sum_{i=1}^{k}x_i\lambda^{i-1}\pm\lambda^{k+1}\right), \tag{7.5}$$

for $k=0,1,2,\cdots,2m$ (the notation $A_2^{A_k^\pm}$ is taken from [14]*).*

Proof. If f has A_1 singularity than it is obvious that F is stably $(1,2m)$-$\mathcal{R}^+$-equivalent to the trivial unfolding. Now we assume that f has A_2 singularity. Since F is stable than F is stable $(1,2m)$-$\mathcal{R}^+$-equivalent to $F(\lambda,x,t)=t^3+tg(\lambda,x)$, where g is a smooth function-germ vanishing at 0. If g is a versal unfolding of the function-germ $\lambda\mapsto g(\lambda,0)$ with A_k singularity we can reduce F to the form (7.5) by a diffeomorphism-germ of the form $(\lambda,x,t)\mapsto(\Lambda(\lambda,x),X(x),t)$. The following lemma shows that these are the only $(1,2m)$-$\mathcal{R}^+$-stable unfoldings. □

Lemma 7.13. *Unfoldings of $A_3^\pm$ singularity are not $(1,2m)$-$\mathcal{R}^+$-stable.*

Proof. If f has A_3 singularity then F is stable $(1,2m)$-$\mathcal{R}^+$-equivalent to $F(\lambda,x,t)=\pm t^4+t^2g_2(\lambda,x)+tg_1(\lambda,x)$, where g_1,g_2 are smooth function-germs vanishing at 0. Now we use the standard arguments of

the singularity theory that stability implies infinitesimal stability. In the case of $(1,2m)$-$\mathcal{R}^+$-equivalence the infinitesimal stability implies the following condition:
(7.6)
$$\mathcal{E}_2 = \mathcal{E}_2\left\langle \frac{\partial F}{\partial t}|_{\mathbb{R}^2}\right\rangle + \mathcal{E}_1\left\langle 1, \frac{\partial F}{\partial \lambda}|_{\mathbb{R}^2}\right\rangle + \mathbb{R}\left\langle \frac{\partial F}{\partial x_1}|_{\mathbb{R}^2}, \cdots, \frac{\partial F}{\partial x_{2m}}|_{\mathbb{R}^2}\right\rangle + \mathfrak{m}_2^{2m+4},$$
where $\mathbb{R}^2$ denotes the t,λ-plane $\{x=0\}$, $\mathcal{E}_2$ is the ring of smooth function-germs in λ and t, $\mathfrak{m}_2$ is the maximal ideal in $\mathcal{E}_2$ and $\mathcal{E}_1$ is the ring of smooth function-germs in λ. Now we use the method from [19].

Let $V = \mathcal{E}_2 / \left(\mathcal{E}_2\left\langle \frac{\partial F}{\partial t}|_{\mathbb{R}^2}\right\rangle + \mathfrak{m}_2^{2m+4}\right)$ and let $\pi : \mathcal{E}_2 \to V$ be the projection. We have $\pi(t^3) = \pi(\mp 1/2tg_2|_{\mathbb{R}^2} \mp 1/4g_1|_{\mathbb{R}^2})$ in V. Thus elements $\pi(t^i\lambda^j)$ for $i=0,1,2$ and $j < 2m+4-i$ form a basis of V over $\mathbb{R}$. It implies that $\dim_{\mathbb{R}} V = 6m+9$. Moreover $\frac{\partial F}{\partial \lambda}|_{\mathbb{R}^2} = t\left(t\frac{\partial g_2}{\partial \lambda}|_{\mathbb{R}^2} + \frac{\partial g_2}{\partial \lambda}|_{\mathbb{R}^2}\right)$. Then
$$\dim_{\mathbb{R}} \pi\left(\mathcal{E}_1\left\langle 1, \frac{\partial F}{\partial \lambda}|_{\mathbb{R}^2}\right\rangle\right) \leq 4m+7$$
and
$$\dim_{\mathbb{R}} \pi\left(\mathbb{R}\left\langle \frac{\partial F}{\partial x_1}|_{\mathbb{R}^2}, \cdots, \frac{\partial F}{\partial x_{2m}}|_{\mathbb{R}^2}\right\rangle\right) \leq 2m.$$
So if (7.6) held we would have $\dim_{\mathbb{R}} V \leq 6m+7 < 6m+9$, which is impossible. Therefore F is not $(1,2m)$-$\mathcal{R}^+$-stable and A_3 singularity has no $(1,2m)$-$\mathcal{R}^+$-stable unfoldings. □

To study the Wigner caustic in the GCS set we consider the germ of F at $(1/2, a, \kappa_a)$.

Theorem 7.14. *If F is the germ at $(\frac{1}{2}, a, \kappa_a)$ of a $(1,2m)$-$\mathcal{R}^+$-stable unfolding of $f \in \mathfrak{m}^2$ then F is stably $(1,2m)$-$\mathcal{R}^+$-equivalent (for $\lambda = 1/2$) to the germ of the trivial unfolding (if f has A_1 singularity) or to one of the following germs at $(\frac{1}{2}, 0, 0)$ of unfoldings of $f(t) = t^3$*

(7.7) $$A_2^{B_k^\pm} \ : \ F(\lambda, x, t) = t^3 + t\left(\sum_{i=0}^{k-1} x_{i+1}\left(\lambda - \frac{1}{2}\right)^i \pm \left(\lambda - \frac{1}{2}\right)^k\right),$$

for $k = 1, 2, \cdots, 2m$ (the notation $A_2^{B_k^\pm}$ is taken from [14]*).*

Proof. If f has A_1 singularity than it is obvious that F is stably $(1,2m)$-$\mathcal{R}^+$-equivalent to the trivial unfolding. Now we assume that f has A_2 singularity. Since F is stable than F is stable $(1,2m)$-$\mathcal{R}^+$-equivalent to $F(\lambda, x, t) = t^3 + tg(\lambda, x)$, where g is a smooth function-germ vanishing at $(1/2, 0)$. If g is a versal unfolding of the function-germ $\lambda \mapsto g(\lambda, 0)$ with $B_k^\pm$ singularity on a manifold (λ-space) with the boundary ($\lambda = \frac{1}{2}$)

(see [1]) then we can reduce F to the form (7.7) by a diffeomorphism-germ of the form $(\lambda, x, t) \mapsto (1/2 + (\lambda - 1/2)\Lambda(\lambda, x), X(x), t)$. $\square$

Theorem 7.15. *If the generating family F for $\mathcal{L}$ has $A_2^{A_k^\pm}$ singularity, for $k = 0, 1, 2, \cdots, 2m$, then $\mathbb{E}(L)$ is a germ of a smooth hypersurface in $\mathbb{R} \times \mathbb{R}^{2m}$.*

If F has $A_2^{A_0}$ singularity at (λ_a, a, κ_a) then $\mathbb{E}(L)$ is transversal at (λ_a, a) to the fibers of projection π.

If F has $A_2^{A_k^\pm}$ singularity for $k \geq 1$ at (λ_a, a, κ_a) then $\mathbb{E}(L)$ is k-tangent at (λ_a, a) to the fibers of projection π, a belongs to the criminant $\Delta(L)$ of $GSC(L)$ and the germ of $\Delta(L)$ at a is the caustic of $A_k^\pm$ singularity.

Proof. By Proposition 6.1 and the normal form of F for $A_2^{A_k^\pm}$ singularity we obtain that $\mathbb{E}(L) = \{(\lambda, x) \in \mathbb{R} \times \mathbb{R}^{2m} : \sum_{i=1}^{k} x_i \lambda^{i-1} \pm \lambda^{k+1} = 0\}$.

It is easy to see that $\mathbb{E}(L)$ is the germ at $(0, 0)$ of a smooth hypersurface and $\mathbb{E}(L)$ is transversal at $(0, 0)$ to $\{\lambda = 0\}$ for $k = 0$ and $\mathbb{E}(L)$ is k-tangent to $\{\lambda = 0\}$ at $(0, 0)$ for $k = 1, 2, \cdots, 2m$. The germ of the criminat $\Delta(L)$ at 0 is described in the following way

$$\{x \in \mathbb{R}^{2m} : \exists\lambda \ \sum_{i=1}^{k} x_i \lambda^{i-1} \pm \lambda^{k+1} = 0, \ \sum_{i=2}^{k} (i-1) x_i \lambda^{i-2} \pm (k+1)\lambda^k = 0\}.$$

So $\Delta(L)$ is a caustic of $A_k^\pm$ singularity. $\square$

Theorem 7.16. *If the germ at $(\frac{1}{2}, a, \kappa_a)$ of a generating family F for $\mathcal{L}$ has $A_2^{B_k^\pm}$ singularity, for $k = 1, 2, \cdots, 2m$, then $\mathbb{E}(L)$ is a germ of a smooth hypersurface in $\mathbb{R} \times \mathbb{R}^{2m}$.*

If F has $A_2^{B_1}$ singularity at $(\frac{1}{2}, a, \kappa_a)$ then $\mathbb{E}(L)$ is transversal at $(\frac{1}{2}, a)$ to the fibers of projection π. The germ of $GCS(L)$ at a is the germ of a smooth hypersurface of $\mathbb{R}^{2m}$ - the Wigner caustic $E_{\frac{1}{2}}(L)$.

If F has $A_2^{B_k^\pm}$ singularity for $k \geq 2$ at $(\frac{1}{2}, a, \kappa_a)$ then $\mathbb{E}(L)$ is k-tangent at $(1/2, a, t)$ to the fibers of projection π. The germ of $GCS(L)$ at a consists of two tangent components: the germ of a smooth hypersurface - the Wigner caustic $E_{\frac{1}{2}}(L)$ and the germ of the caustic of $B_k^\pm$ singularity - the criminant $\Delta(L)$.

Proof. By Proposition 6.1 and the normal form of F for $A_2^{B_k^\pm}$ singularity we obtain that

$$\mathbb{E}(L) = \{(\lambda, x) \in \mathbb{R} \times \mathbb{R}^{2m} : \sum_{i=0}^{k-1} x_{i+1}(\lambda - 1/2)^i \pm (\lambda - 1/2)^k = 0\}.$$

The Wigner caustic $E_{1/2}(L) = \{x \in \mathbb{R}^{2m} : x_1 = 0\}$ is the germ of a smooth hypersurface.

It is easy to see that $\mathbb{E}(L)$ is the germ at $(1/2, 0)$ of a smooth hypersurface and $\mathbb{E}(L)$ is transversal at $(1/2, 0)$ to $\{\lambda = 1/2\}$ for $k = 1$.

$\mathbb{E}(L)$ is k-tangent to $\{\lambda = 1/2\}$ at $(1/2, 0)$ for $k = 2, \cdots, 2m$. The germ of the criminat $\Delta(L)$ at 0 is described in the following way

$$\{x \in \mathbb{R}^{2m} : \exists\tau \ \sum_{i=0}^{k-1} x_{i+1}\tau^i \pm \tau^k = 0, \ \sum_{i=1}^{k-1} i x_{i+1}\tau^{i-1} \pm k\tau^{k-1} = 0\}.$$

So $\Delta(L)$ is a caustic of $B_k^{\pm}$ singularity and $E_{1/2}(L)$ is tangent to $\Delta(L)$ at 0. □

Remark 7.17. Not all $(1, 2m)$-$\mathcal{R}^+$-stable singularities can be realizable as singularities of generating families F for $\mathcal{L}$ which are of the special form given in Theorem 4.8. In the next section, in Theorem 8.7, we prove that the $A_2^{A_2}$ singularity is not realizable for Lagrangian curves.

8. Classifications of the GCS of Lagrangian curves

In this section, using the equivalence of $GCS(L)$ introduced in section 7, we classify the singularities of the Global Centre Symmetry set of a Lagrangian curve L, that is, a curve $L \subset (\mathbb{R}^2, \omega)$.

To set the stage, we first state the results for the GCS of a curve on the affine plane $\mathbb{R}^2$, when no symplectic structure on $\mathbb{R}^2$ is considered.

The main results for this non-Lagrangian case were obtained in [3], [15] and [9]-[10] by various methods and are summarized in Theorem 8.1 below, which can also be proved using the affine-invariant method of chord equivalence, which is the analogous of $(1, 2m)$-Lagrangian equivalence when no symplectic structure is considered.

Theorem 8.2 presents global results for the GCS of a convex curve, some of which have not been stated before.

Theorem 8.1 ([3], [15], [9]-[10])**.** *Affine stable GCS of a smooth convex closed curve $M \subset \mathbb{R}^2$ (no symplectic structure) consists of three components:*
i) The CSS, a smooth curve with (possible) self intersections and cusp singularities, ii) the Wigner caustic, a smooth curve with (possible) self intersections and cusp singularities lying on the smooth part of the CSS, and iii) the middle axes, which are smooth half-lines starting at the the cusp points of the CSS.

In Theorem 8.1, the CSS and the middle axes form, together, the centre symmetry caustic $\Sigma'(M)$.

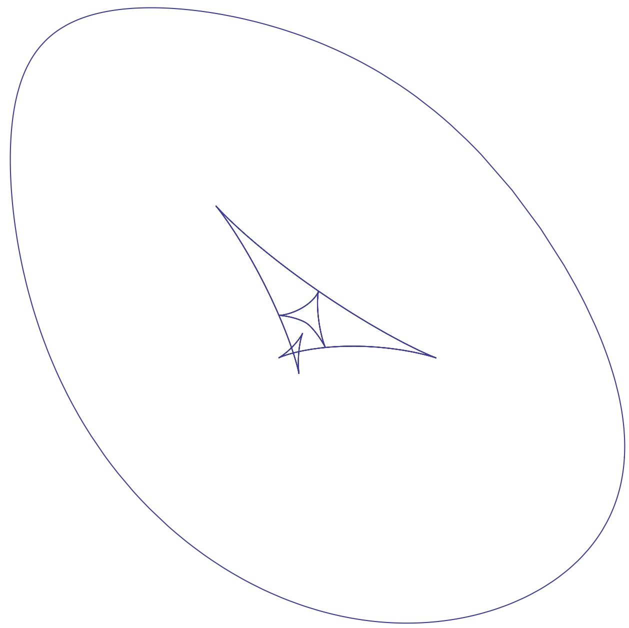

Figure 1. GCS of an oval in nonsymplectic plane: CSS with 5 cusps and Wigner caustic with 3 cusps (the middle axes are not shown here).

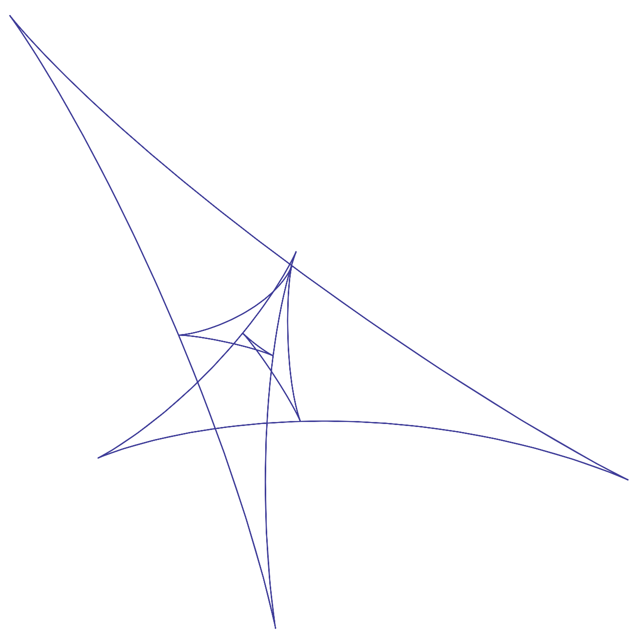

Figure 2. Both the CSS and the Wigner caustic with five cusps.

Theorem 8.2. *Let M be a generic smooth convex closed curve in $\mathbb{R}^2$. The number of cusps of the Wigner caustic of M is odd and not smaller than 3. The number of cusps of the CSS of M is odd and not smaller than 3. The number of cusps of the Wigner caustic of M is not greater than the number of cusps of the CSS of M.*

Proof. The first statement, on the number of cusps of Wigner caustics, was first proven by Berry [3] and the second statement, on the number of cusps of CSS, was first proven by Giblin and Holtom [8]. The last inequality follows immediately from the characterization in [8] of cusps of $E_{1/2}(M)$ by the curvature ratio being 1 and cusps of CSS of M by the derivative of the curvature ratio being 0, and from Rolle's theorem. □

Figures of $GCS(M)$ where the number of cusps of the CSS and of the Wigner caustic are equal to three and neither curve is self intersecting can be found in [8]. We have pictured a case when the number of cusps of the Wigner caustic is three and the CSS is self intersecting and the number of its cusps is five (Fig. 1), and another case when both the Wigner caustic and the CSS are self intersecting and each one has five cusps (Fig. 2).

8.1. **Affine symplectic invariant classification of GCS of Lagrangian curves.** Let L be a smooth closed (Lagrangian) curve in the symplectic affine space $(\mathbb{R}^2, \omega = dp \wedge dq)$. Using the $(1,2)$-Lagrangian equivalence introduced in the previous section (Definition 7.3), we classify the singularities of $GCS(L)$.

Let $a^+ = (p_a^+, q_a^+), a^- = (p_a^-, q_a^-) \in L$ be a parallel pair on L and $a_\lambda = \lambda a^+ + (1-\lambda)a^-$, $\dot{q}_\lambda = \lambda q_a^+ - (1-\lambda)q_a^-$. Let $S^\pm$ be germs of generating functions of L at $a^\pm$ satisfying the conditions in Proposition 4.7. Then the germ of generating family of $\mathcal{L}$ has the following form

$$F(\lambda, p, q, t) = 2\lambda^2 S^+(\frac{q+t}{2\lambda}) - 2(1-\lambda)^2 S^-(\frac{q-t}{2(1-\lambda)}) - pt.$$

The big front is described in the following way

$$\mathbb{E}(L) = \left\{ (\lambda, p, q) \in \mathbb{R} \times \mathbb{R}^2 : \exists t \ \frac{\partial F}{\partial t}(\lambda, p, q, t) = \frac{\partial^2 F}{\partial t^2}(\lambda, p, q, t) = 0 \right\}.$$

In the following propositions we present descriptions of different positions of $\mathbb{E}(L)$ with respect to the fiber bundle π in terms of the generating family F, generating functions S^+ and S^- and their geometric interpretations.

Proposition 8.3. *The following conditions are equivalent*

(i) (λ, a_λ) *belongs the regular part of* $\mathbb{E}(L)$,

(ii) $\exists t\ \frac{\partial^3 F}{\partial t^3}(\lambda, a_\lambda, t) \neq 0, \frac{\partial F}{\partial t}(\lambda, a_\lambda, t) = \frac{\partial^2 F}{\partial t^2}(\lambda, a_\lambda, t) = 0,$
(iii) $\frac{1}{\lambda}\frac{\partial^3 S^+}{\partial (q^+)^3}(q_a^+) + \frac{1}{1-\lambda}\frac{\partial^3 S^-}{\partial (q^-)^3}(q_a^-) \neq 0,$
(iv) $\frac{1}{\lambda}\kappa(a^+) + \frac{1}{1-\lambda}\kappa(a^-) \neq 0$, *where* $\kappa(x)$ *is the curvature of* L *at* x.

Proof. Equivalence of (i) and (ii) follows from the definition of the regular part of $\mathbb{E}(L)$. Equivalence of (ii) and (iii) is obtained by direct calculations. (iv) is obvious since $\kappa(a^\pm) = \frac{\partial^3 S^\pm}{\partial (q^\pm)^3}(q_a^\pm)$. □

Proposition 8.4. *The following conditions are equivalent*

(v) *the regular part of* $\mathbb{E}(L)$ *is tangent to the fiber of* π *at* (λ, a_λ),
(vi) $\exists t$ *(ii) is satisfied and* $\frac{\partial^2 F}{\partial \lambda \partial t}(\lambda, a_\lambda, t) = 0$.
(vii) *(iii) is satisfied and* $p_a^+ = \frac{\partial S^+}{\partial q^+}(q_a^+) = \frac{\partial S^-}{\partial q^-}(q_a^-) = p_a^-$.
(viii) *(iv) is satisfied and* $l(a^+, a^-)$ *is bitangent to* a^+, a^- *to* L.

Proof. All statements follow from Proposition 6.4 and Theorem 6.5. □

Proposition 8.5. *The following conditions are equivalent*

(ix) *the regular part of* $\mathbb{E}(L)$ *is* 1*-tangent to the fiber of* π *at* (λ, a_λ),
(x) $\exists t$ *(vi) is satisfied and*

$$\left(\frac{\partial^3 F}{\partial \lambda \partial t^2}(\lambda, a_\lambda, t)\right)^2 - \frac{\partial^3 F}{\partial t^3}(\lambda, a_\lambda, t)\frac{\partial^3 F}{\partial \lambda^2 \partial t}(\lambda, a_\lambda, t) \neq 0. \tag{8.1}$$

(xi) *(vii) is satisfied and* $\frac{\partial^3 S^+}{\partial (q^+)^3}(q_a^+)\frac{\partial^3 S^-}{\partial (q^-)^3}(q_a^-) \neq 0$.
(xii) *(iv) is satisfied and* $l(a^+, a^-)$ *is* 1*-tangent to* L *at* a^+ *and* a^-

Proof. (λ, a_λ) is a regular point of $\mathbb{E}(L) = \left\{(\lambda, p, q) : \exists t\ \frac{\partial F}{\partial t} = \frac{\partial^2 F}{\partial t^2} = 0\right\}$. By Proposition 8.3 it means that $\frac{\partial^3 F}{\partial t^3}(\lambda, a_\lambda, t) \neq 0$. It implies that there exists a smooth function-germ T on $\mathbb{R}^3$ such that $\frac{\partial^2 F}{\partial t^2}(\lambda, p, q, t) = 0$ iff $t = T(\lambda, p, q)$. Then $\mathbb{E}(L) = \left\{(\lambda, p, q) : \frac{\partial F}{\partial t}(\lambda, p, q, T(\lambda, p, q)) = 0\right\}$. Then (ix) is equivalent to

$$\frac{\partial}{\partial \lambda}\left(\frac{\partial F}{\partial t}(\lambda, p, q, T(\lambda, p, q))\right)\Big|_{(\lambda, a_\lambda)} = 0 \tag{8.2}$$

$$\frac{\partial^2}{\partial \lambda^2}\left(\frac{\partial F}{\partial t}(\lambda, p, q, T(\lambda, p, q))\right)\Big|_{(\lambda, a_\lambda)} \neq 0. \tag{8.3}$$

Using the formulae

(8.4)
$$\frac{\partial T}{\partial \lambda}(\lambda, p, q) = -\left(\frac{\partial^2 F}{\partial t^3}(\lambda, p, q, T(\lambda, p, q)\right)^{-1}\frac{\partial^2 F}{\partial \lambda \partial t^2}(\lambda, p, q, T(\lambda, p, q))$$

it is easy to check that (8.2)-(8.3) are equivalent to (x). Equivalence of (x) and (xi) is obtained by direct calculation and the last equivalence is obvious. □

Proposition 8.6. *The following conditions are equivalent*

(xiii) *the regular part of* $\mathbb{E}(L)$ *is* 2*-tangent to the fiber of* π *at* (λ, a_λ),

(xiv) $\exists t$ *(vi) is satisfied, (8.1) is not satisfied and*

$$\Big(\frac{\partial^4 F}{\partial\lambda^3\partial t}\left(\frac{\partial^3 F}{\partial t^3}\right)^3 - 3\frac{\partial^4 F}{\partial\lambda^2\partial t^2}\left(\frac{\partial^3 F}{\partial t^3}\right)^2\frac{\partial^3 F}{\partial\lambda\partial t^2}+$$

$$+3\frac{\partial^4 F}{\partial\lambda\partial t^3}\frac{\partial^3 F}{\partial t^3}\left(\frac{\partial^3 F}{\partial\lambda\partial t^2}\right)^2 - \frac{\partial^4 F}{\partial t^4}\left(\frac{\partial^3 F}{\partial\lambda\partial t^2}\right)^3\Big)(\lambda, a_\lambda, t) \neq 0.$$

(xv) *(vii) is satisfied,*

$$\frac{\partial^3 S^+}{\partial(q^+)^3}(q_a^+) = 0 \ \wedge\ \frac{\partial^4 S^+}{\partial(q^+)^4}(q_a^+) \neq 0$$

or

$$\frac{\partial^3 S^-}{\partial(q^-)^3}(q_a^-) = 0 \ \wedge\ \frac{\partial^4 S^-}{\partial(q^-)^4}(q_a^-) \neq 0.$$

(xvi) *(iv) is satisfied and* $l(a^+, a^-)$ *is* 1*-tangent to* L *at one of points* a^+, a^- *and* 2*-tangent to* L *at the other.*

Proof. We use the same notation as in the proof of Proposition 8.5. (xiii) means that (8.2) is satisfied, (8.3) is not satisfied and

$$\frac{\partial^3}{\partial\lambda^3}\left(\frac{\partial F}{\partial t}(\lambda, p, q, T(\lambda, p, q))\right)\Big|_{(\lambda, a_\lambda)} \neq 0. \tag{8.5}$$

Using (8.4) it is easy to check that these conditions are equivalent to (xiv). By direct calculation one can obtain that (xiv) is equivalent to (xv) and (xvi) is obvious geometric description of (xv). □

Theorem 8.7. *Let* $\frac{1}{\lambda}\frac{\partial^3 S^+}{\partial(q^+)^3}(q_a^+) + \frac{1}{1-\lambda}\frac{\partial^3 S^-}{\partial(q^-)^3}(q_a^-) \neq 0$ *(for statements (1)-(2) below,* $\lambda = 1/2$*).*

(1) *If the chord* $l(a^+, a^-)$ *is not bitangent to* L *at* a^+, a^- *then the germ of* F *at* $(1/2, a_{1/2}, \dot{q}_{1/2})$ *has* $A_2^{B_1}$ *singularity and the germ of* GCS *at* $a_{1/2}$ *is a smooth curve (the smooth part of the Wigner caustic).*

(2) *If the chord* $l(a^+, a^-)$ *is* 1*-tangent to* L *at* a^+ *and at* a^- *then the germ of* F *at* $(1/2, a_{1/2}, \dot{q}_{1/2})$ *has* $A_2^{B_2}$ *singularity and the germ of* GCS *at* $a_{1/2}$ *is a union of two* 1*-tangent smooth curves (the smooth part of the Wigner caustic and the smooth part of the criminant).*

(3) *If the chord $l(a^+,a^-)$ is 1-tangent to L at a^+ and at a^- then the germ of F at $(\lambda, a_\lambda, \dot{q}_\lambda)$ for $\lambda \neq 1/2$ has $A_2^{A_1}$ singularity and the germ of GCS at a_λ is a smooth curve (the smooth part of the criminant).*

(4) *If the chord $l(a^+,a^-)$ is 1-tangent to L at one of the points a^+, a^- and 2-tangent to L at the other point then the germ of F at $(\lambda, a_\lambda, \dot{q}_\lambda)$ for $\lambda \neq 1/2$ is not $(1,2)$-$\mathcal{R}^+$-stable. $A_2^{A_2}$ is not realizable as a singularity of GCS of a Lagrangian curve.*

Proof. By Proposition 8.3 if

$$\frac{1}{\lambda}\frac{\partial^3 S^+}{\partial (q^+)^3}(q_a^+) + \frac{1}{1-\lambda}\frac{\partial^3 S^-}{\partial (q^-)^3}(q_a^-) \neq 0 \tag{8.6}$$

then the germ of a generating family F of $\mathcal{L}$ is a unfolding of the function-germ with A_2 singularity. Therefore we can reduce F to the following form $F'(\lambda,p,q,t) = t^3 + g(\lambda,p,q)t$, where g is a smooth function-germ vanishing at $(\lambda_a, 0)$ (for $\lambda_a = 0$ or $\lambda_a = 1/2$).

By Proposition 8.4 if the chord $l(a^+,a^-)$ is not bitangent to L at a^+, a^- then $\frac{\partial F'}{\partial t \partial \lambda}(1/2,0,0) \neq 0$ and this implies that $\frac{\partial g}{\partial \lambda}(1/2,0) \neq 0$. By Theorems 7.14 and 7.16 we obtain (1).

If the chord $l(a^+,a^-)$ is tangent to L at a^+, a^- then by Proposition 8.4 we get that $p_a^+ = p_a^-$ and $\frac{\partial F'}{\partial t \partial \lambda}(\lambda_a,0,0) = 0$ and this implies that $\frac{\partial g}{\partial \lambda}(\lambda_a,0) = 0$. But $dg|_{(\lambda_a,0)} \neq 0$ since $\frac{\partial F}{\partial t \partial p}(\lambda_a, a, \dot{q}_a) \neq 0$.

By Proposition 8.5 if $l(a^+,a^-)$ is 1-tangent to L at a^+, a^- then

$$\left(\frac{\partial^3 F'}{\partial \lambda \partial t^2}(\lambda_a,0,0)\right)^2 - \frac{\partial^3 F'}{\partial t^3}(\lambda_a,0,0)\frac{\partial^3 F'}{\partial \lambda^2 \partial t}(\lambda_a,0,0) \neq 0. \tag{8.7}$$

But this implies that $\frac{\partial^2 g}{\partial \lambda^2}(\lambda_a, 0,) \neq 0$. Thus if $\lambda_a = 1/2$ by Theorems 7.14 and 7.16 we obtain (2) and otherwise by Theorems 7.12 and 7.15 we obtain (3).

Finally, let us assume that the chord $l(a^+,a^-)$ is 1-tangent to L at a^+ and 2-tangent at a^-. By Proposition 8.6 we get $\frac{\partial^2 g}{\partial \lambda^2}(\lambda_a, 0,) = 0$ and

$$(\frac{\partial^4 F}{\partial \lambda^3 \partial t}\left(\frac{\partial^3 F}{\partial t^3}\right)^3 - 3\frac{\partial^4 F}{\partial \lambda^2 \partial t^2}\left(\frac{\partial^3 F}{\partial t^3}\right)^2 \frac{\partial^3 F}{\partial \lambda \partial t^2} +$$

$$+3\frac{\partial^4 F}{\partial \lambda \partial t^3}\frac{\partial^3 F}{\partial t^3}\left(\frac{\partial^3 F}{\partial \lambda \partial t^2}\right)^2 - \frac{\partial^4 F}{\partial t^4}\left(\frac{\partial^3 F}{\partial \lambda \partial t^2}\right)^3)(\lambda_a,0,0) \neq 0.$$

Thus, $\frac{\partial^3 g}{\partial \lambda^3}(\lambda_a, 0,) \neq 0$. We know that $\frac{\partial g}{\partial p}(\lambda_a, 0,) \neq 0$ since $\frac{\partial^2 F}{\partial t \partial p}(\lambda_a, a, \dot{q}_a) \neq 0$. It is easy to see that $\frac{\partial^2 F}{\partial t \partial q}(\lambda_a, a, \dot{q}_a) = 0$. Thus F has $A_2^{A_2}$ singularity

at $(\lambda_a, a, \dot{q}_a)$ iff the following condition is satisfied

$$\frac{\partial^3 F}{\partial\lambda\partial q\partial t}(\lambda_a, a, \dot{q}_a)\frac{\partial^3 F}{\partial t^3}(\lambda_a, a, \dot{q}_a) - \frac{\partial^3 F}{\partial\lambda\partial t^2}(\lambda_a, a, \dot{q}_a)\frac{\partial^3 F}{\partial q\partial t^2}(\lambda_a, a, \dot{q}_a) \neq 0$$

By direct calculation it is easy to see that this is equivalent to

$$\frac{(q_a^+ - q_a^-)}{\lambda_a(1-\lambda_a)}\frac{\partial^3 S^+}{\partial(q^+)^3}(q_a^+)\frac{\partial^3 S^-}{\partial(q^-)^3}(q_a^-) \neq 0,$$

which is not satisfied, since $l(a^+, a^-)$ is 2-tangent to L at a^-. $\square$

Corollary 8.8. *Let L be a smooth closed convex curve in $(\mathbb{R}^2, \omega)$. The middle axes and the whole CSS are not $(1,2)$-Lagrangian stable. The smooth part of the Wigner caustic is $(1,2)$-Lagrangian stable, but the cusp singularities of the Wigner caustic, seen as part of the $GCS(L)$, are not $(1,2)$-Lagrangian stable.*

Remark 8.9. A comparison of Theorem 8.1 and Corollary 8.8 shows that, for the case of convex curves in $\mathbb{R}^2$, various singularities which are affine stable are not affine-Lagrangian stable. In other words, there is a breakdown of stability of various singularities due to the presence of a symplectic form in $\mathbb{R}^2$ to be accounted for. Other cases of breakdown of simplicity due to a symplectic form can be found in [4, 6, 7]. Note also that, although the cusp singularities of the Wigner caustic are affine-Lagrangian stable when the Wigner caustic is considered by itself (Corollary 5.2), they are not affine-Lagrangian stable when the Wigner caustic is considered as part of the GCS. That is, the meeting of the Wigner caustic and the CSS is not affine-Lagrangian stable.

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